Rapidly Converging Low-Complexity Iterative Transmit Precoders for Massive MIMO Downlink

Zheng Wang, *Senior Member, IEEE*, Jiaheng Wang, *Senior Member, IEEE*, Zhen Gao, Yongming Huang, *Senior Member, IEEE*, Derrick Wing Kwan Ng, *Fellow, IEEE*, and Lajos Hanzo, *Life Fellow, IEEE*

***Abstract*—In this paper, rapidly converging low-complexity iterative transmit precoding (TPC) techniques are proposed for the massive multiple-input multiple-output (MIMO) downlink. First of all, the proposed random block-based iterative TPC (RBI-TPC) algorithm performs its iterations by updating multiple rather than a single component at each instant, where the updating order of each block containing multiple components relies on the samples randomly sampled from a discrete distribution. Based on the analytically derived convergence rate, we demonstrate that improved convergence is achieved by the block-based update mechanism conceived since the correlation between multiple components can be beneficially exploited. Then, the random sampling that determines the updating order is studied. By applying conditional random sampling, the updating order is optimized based on the latest updates for attaining more rapid convergence. We also demonstrate that the associated updating order may become deterministic under specific conditions so that a fixed but optimized updating order can be used for facilitating the practical implementations, which paves the way for conceiving the ordered block-based iterative TPC (OBI-TPC) algorithm. Finally, the concept of successive over-relaxation (SOR) is adopted for further convergence improvement and simulations are presented to illustrate the performance improvements of the proposed RBI and OBI TPC algorithms compared to the existing low-complexity iterative TPC schemes.**

***Index Terms*—Massive MIMO, low-complexity linear transmit precoding, iterative methods, random sampling.**

I. Introduction

AS ONE of the core technologies of next-generation wireless communications, massive multiple-input multiple-output (MIMO) solutions substantially improve the spectral efficiency and power efficiency trade-off [1], [2], [3], [4], [5]. However, their performance is eroded by the multi-user interference in the downlink [6], [7], [8], [9]. As an efficient technique of mitigating the interference, transmit precoding (TPC) has been widely applied for targeting the transmitted signal at the intended receiver, which mitigates the interference imposed on other users [10], [11], [12], [13], [14], [15], [16]. It has been demonstrated that capacity-approaching performance can be achieved by linear TPC techniques including zero-forcing (ZF) and regularized ZF (RZF) TPCs when the number of antennas N used at base station (BS) is sufficiently higher than that employed by the receiver (denoted by K) [17], [18].

Unfortunately, due to the matrix inversion associated with a complexity order of $O(K^3)$, both the ZF and RZF TPCs tend to have excessive complexity cost. Hence, a number of low-complexity TPC schemes have been proposed for implementing the matrix inversion by polynomial expansion or iterative methods [19], [20], [21], [22], [23], [24], [25]. Specifically, the Neumann series (NS) and Kapteyn series (KS) based on polynomial expansion are applied for linear TPC in [26], [27]. Compared to polynomial expansion, approximating the matrix inversion by iterative methods is more promising as a benefit of its faster convergence. In [28], the Jacobi iteration is employed for determining the linear TPC coefficients at a low complexity. However, the convergence of Jacobi iteration is only guaranteed when $N \gg K$, making it limited to various scenarios of interest. In [29], the Richardson iteration is applied for linear TPC with an introduced relaxation factor ω. To improve the TPC performance, Gauss-Seidel (GS) iteration is applied in [30], which results in a faster convergence than Jacobi and Richardson iterations. Moreover,

Manuscript received 31 October 2022; revised 7 April 2023 and 25 June 2023; accepted 7 August 2023. Date of publication 15 August 2023; date of current version 19 December 2023. This work was supported in part by the National Natural Science Foundation of China under Grant 61801216, 62371124, 61720106003, 62225107, 61971130 and U22B2006, in part by the National Key R&D Program of China under Grant 2018YFB1800801, in part by the Natural Science Foundation of Jiangsu Province under Grant BK20180420, in part by the Natural Science Foundation on Frontier Leading Technology Basic Research Project of Jiangsu under Grants BK20222001 and BK20212001, in part by the Key Technologies R&D Program of Jiangsu (Prospective and Key Technologies for Industry) under Grants BE2022068-3 and BE2022067-5, in part by the Fundamental Research Funds for the Central Universities 2242022k60002 and 2242023k5003. Derrick Wing Kwan Ng is supported by the Australian Research Council's Discovery Projects (DP210102169, DP230100603). Lajos Hanzo would like to acknowledge the financial support of the Engineering and Physical Sciences Research Council projects EP/W016605/1, EP/X01228X/1 and EP/X01228X/1 as well as of the European Research Council's Advanced Fellow Grant QuantCom (Grant No. 789028). The associate editor coordinating the review of this article and approving it for publication was N. Lee. *(Corresponding authors: Lajos Hanzo; Zheng Wang.)*

Zheng Wang, Jiaheng Wang, and Yongming Huang are with the National Mobile Communications Research Laboratory, School of Information Science and Engineering, and the Frontiers Science Center for Mobile Information Communication and Security, Southeast University, Nanjing 210096, China (e-mail: wznuaa@gmail.com).

Zhen Gao is with the School of Information and Electronics, Beijing Institute of Technology, Beijing 100081, China (e-mail: gaozhen16@bit.edu.cn).

Derrick Wing Kwan Ng is with the School of Electrical Engineering and Telecommunications, University of New South Wales, Sydney, NSW 2052, Australia (e-mail: w.k.ng@unsw.edu.au).

Lajos Hanzo is with the School of Electronics and Computer Science, University of Southampton, SO17 1BJ Southampton, U.K. (e-mail: lh@ecs.soton.ac.uk).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TCOMM.2023.3305646.

Digital Object Identifier 10.1109/TCOMM.2023.3305646

by incorporating a relaxation factor into the GS iteration, it has been shown that successive over-relaxation (SOR) iteration converges more rapid than GS iteration [31]. A range of other low-complexity iterative TPC algorithms can be found in [32], [33], [34], and [35].

In contrast to the Jacobi and Richardson iterations that allow parallel updates during the iterations, the GS iteration updates each element based on the latest update of all the other elements [36], which leads to a sequential structure. Therefore, to achieve an improved precoding performance for massive MIMO systems, a pair of substantial improvements are conceived in this paper with respect to the iteration-based linear TPC. The first improvement is based on updating multiple elements rather than single element at each instant. By performing such a block-based update, the correlation between multiple elements can be beneficially exploited, which may lead to further convergence improvement. The second one is concerned with the update order of the iteration. In contrast to the forward or backward update order by default [18], considerable convergence gains may be attained by a tailor-made update order.

It has been shown in [37] that the convergence behaviour of the iterative methods may be analyzed statistically by means of random sampling. Indeed, random sampling has emerged as a powerful mathematical tool capable of addressing various challenging problems of interest [38], [39], [40], [41], [42], [43], [44]. Inspired by this success, we carry out the associated convergence analysis of the iteration by harnessing random sampling from specifically a designed distribution. Then, based on the convergence rate derived, the performance gains attained by the block-based update and our bespoke update order can be clearly demonstrated. Moreover, the relationship between the traditional and the random sampling-based iteration is also revealed. In particular, we show that the traditional iteration associated with a certain fixed order can be approximated by random sampling based iteration with the aid of multi-step conditional sampling. Therefore, we can seamlessly incorporate the mechanisms of the ordered and the block-based updates into the traditional iterations, thus attaining an improved TPC performance in downlink massive MIMO systems.

In a nutshell, we advance the research of low-complexity iterative TPC on several fronts.

- Firstly, by carefully revisiting the traditional iterative TPC schemes, the random block-based iterative TPC (RBI-TPC) algorithm is proposed, which always converges in an exponential way in terms of the mean square error (MSE). Based on the derived convergence rate, the convergence gain of the block-based update can be confirmed but a meritorious choice of the sampling distribution is also beneficial for improving the iterative convergence.
- Secondly, by introducing the multi-step conditional sampling technique into our RBI-TPC algorithm, a deterministic-like TPC can be realized, where the fixed update order is determined by the sampling distribution applied. Inspired by this, the ordered block-based iterative TPC (OBI-TPC) algorithm is proposed and we show that OBI-TPC enjoys a faster convergence than RBI-TPC. Meanwhile, because of the fixed order in OBI-TPC, the operations of random sampling can be avoided, so that considerable potential efficiency gains can also be attained by OBI-TPC.
- Thirdly, a relaxation factor is incorporated into the proposed OBI-TPC scheme for achieving an improved precoding performance, and this leads to the ordered block-based iterative successive over-relaxation transmit precoding (OBI-SOR-TPC).

To sum up, we boldly and explicitly contrast our new contributions to the related literature in Table I, where k denotes the iteration index and $\mu(\mathbf{A})$ is the spectral radius of matrix $\mathbf{A}$.

The rest of this paper is organized as follows. Section II describes the traditional linear TPC conceived for the downlink of massive MIMO systems and briefly reviews the family of low-complexity iterative TPC schemes. In Section III, the traditional iterative TPC schemes are revisited in more detail and the RBI-TPC algorithm is proposed. In Section IV, the convergence analysis of the proposed RBI-TPC is presented to demonstrate its globally exponential convergence, followed by the design options for the sampling distribution. In Section V, by taking advantage of conditional sampling, the RBI-TPC with multi-step conditional sampling is illustrated for improving convergence and efficiency. Based on this, the OBI-TPC algorithm is proposed, which can be further enhanced by introducing a relaxation factor. Finally, simulation results characterizing the proposed RBI-TPC and OBI-TPC algorithms in the downlink of massive MIMO systems are presented in Section VI, and Section VII concludes the paper.

Notation: Matrices and column vectors are indicated by upper and lowercase boldface letters, and the conjugate transpose, inverse, pseudoinverse of a matrix $\mathbf{B}$ by $\mathbf{B}^H, \mathbf{B}^{-1}$, and $\mathbf{B}^\dagger$, respectively. We adopt $\mathbf{b}_i$ or $\mathbf{B}_{:,i}$ for the ith column of matrix $\mathbf{B}$, $b_{i,j}$ for the entry in the ith row and jth column of the matrix $\mathbf{B}$. Let $\langle \mathbf{X}, \mathbf{Y} \rangle_{F(\mathbf{W}^{-1})} \triangleq \text{Tr}(\mathbf{X}^H \mathbf{W}^{-1} \mathbf{Y} \mathbf{W}^{-1})$ stand for the weighting Frobenius inner product, where $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n\times n}$ and $\mathbf{W} \in \mathbb{C}^{n\times n}$ is a symmetric positive definite matrix. Furthermore, $\|\cdot\|_F$ denotes the standard Frobenius norm with the identity matrix $\mathbf{I}$ and $\text{Tr}(\cdot)$ denotes the trace of the input matrix. Finally, the computational complexity is measured by the number of multiplication operations.

II. Preliminaries

In this section, the classic linear TPC designed for the downlink transmission in massive MIMO systems is briefly introduced, followed by the background of the traditional low-complexity iterative TPC schemes.

A. Linear Transmit Precoding in Downlink

Consider a massive MIMO system having N downlink transmit antennas at the base station (BS), which serves K single-antenna user terminals (UTs) $(N \geq K)$. Furthermore, let $\mathbf{s} \in \mathbb{C}^K$ indicate the source information transmitted from the BS to the users. Then the received multi-user signal

TABLE I
A BRIEF COMPARISON OF THE RELATED LITERATURE OF LOW-COMPLEXITY ITERATIVE TPC SCHEMES

TPC schemes	Computational Complexity	Convergence	Block-based Update	Ordered Update
ZF & RZF	$O(K^3)$			
Polynomial Expansion [21]	$O(NK \cdot k)$	if $N \gg K$		
Neumann Series [45]	$O(K^2 \cdot k)$, $k < 3$	if $N \gg K$		
Newton [34]	$O(K^2 \cdot k)$	if $N \gg K$		
Jacobi [28]	$O(K^2 \cdot k)$	if $N \gg K$		
Richardson [29]	$O(K^2 \cdot k)$	if $0 < \omega < 2/\mu(\mathbf{A})$		
Kaczmarz [46]	$O(K^2 \cdot k)$	always		
Gauss Seidel [19]	$O(K^2 \cdot k)$	always		
SOR [31]	$O(K^2 \cdot k)$	if $1 < \omega < 2$		
RIPA [37]	$O(K^2 \cdot k)$	always	✓	
OBI (This work)	$O(K^2 \cdot k)$	always	✓	✓
OBI-SOR (This work)	$O(K^2 \cdot k)$	if $1 < \omega < 2$	✓	✓

$\mathbf{y} \in \mathbb{C}^K$ represented in a vectorial form at the UTs is given by [18] and [19]

$$\mathbf{y} = \sqrt{\varrho}\mathbf{H}^H\mathbf{G}\mathbf{s} + \mathbf{n}. \tag{1}$$

Here, $\mathbf{H} \in \mathbb{C}^{N\times K}$ is the channel matrix, $\mathbf{G} \in \mathbb{C}^{N\times K}$ denotes the transmit precoding (TPC) matrix of the downlink transmission, $\mathbf{n} \in \mathbb{C}^K$ is the additive white Gaussian noise (AWGN) whose entries obey $\mathcal{CN}(0, \sigma^2)$, $\varrho > 0$ indicates the average transmit power at the BS. Due to the power constraint, the source information $\mathbf{s}$ and the TPC matrix $\mathbf{G}$ should be specifically selected by normalizing $\|\mathbf{s}\|^2 = 1$ and $\mathrm{tr}(\mathbf{G}\mathbf{G}^H) = 1$. To be more specific, the optimization problem for seeking the linear TPC matrix $\mathbf{G}$ can be formulated as

$$\min_{\mathbf{G}} \; E_{\mathbf{n},\mathbf{s}}[\|\mathbf{y} - \mathbf{s}\|^2]. \tag{2}$$

With respect to the problem in (2), the linear ZF TPC is designed to mitigate the interference in the downlink transmission, i.e.,

$$\mathbf{G}_{\text{zf}} = \beta\mathbf{H}(\mathbf{H}^H\mathbf{H})^{-1}, \tag{3}$$

which conveys the transmitted signal to the target user by nulling the signal in the directions of the other users. Here, β is a scaling factor used for satisfying the power constraint. Moreover, as an improved TPC based on the mean-square error criterion, the linear regularized ZF (RZF) TPC is proposed, which is formulated as

$$\mathbf{G}_{\text{rzf}} = \beta\mathbf{H}(\mathbf{H}^H\mathbf{H} + \xi\mathbf{I})^{-1}. \tag{4}$$

This is also known as the minimum mean-square error (MMSE) TPC. Here, $\mathbf{I}$ stands for the identity matrix and ξ is a coefficient used for scalar regularization. Theoretically, the RZF TPC operates as the ZF TPC when $\xi = 0$, while it degenerates to the traditional maximal ratio transmission (MRT) scheme if ξ goes to infinity.

Intuitively, according to (3) and (4), both the TPC matrices $\mathbf{G}_{\text{zf}}$ and $\mathbf{G}_{\text{rzf}}$ are characterized by the channel matrix $\mathbf{H}$. More specifically, by letting $\mathbf{A} = \mathbf{H}^H\mathbf{H} \in \mathbb{C}^{K\times K}$ for the ZF TPC (or $\mathbf{A} = \mathbf{H}^H\mathbf{H} + \xi\mathbf{I} \in \mathbb{C}^{K\times K}$ for the RZF TPC), we have

$$\mathbf{G} = \beta\mathbf{H}\mathbf{A}^{-1}, \tag{5}$$

and the computation of $\mathbf{G}$ heavily relies on the matrix inversion of $\mathbf{A}$, which has a computational complexity order of $\mathcal{O}(K^3)$. Clearly, the complexity escalates rapidly with the system dimension. This imposes a pressing challenge on the implementation of the linear TPC in practice.

On the other hand, given matrix $\mathbf{A}$, computing its matrix inversion can be replaced by solving the following linear system

$$\mathbf{A}\mathbf{t} = \mathbf{s} \tag{6}$$

as $\mathbf{t} = \mathbf{A}^{-1}\mathbf{s}$. Then, according to $\mathbf{t} = \mathbf{A}^{-1}\mathbf{s}$, the system model in (1) becomes

$$\begin{aligned}\mathbf{y} &= \sqrt{\varrho}\mathbf{H}^H\mathbf{G}\mathbf{s} + \mathbf{n} = \sqrt{\varrho}\mathbf{H}^H\beta\mathbf{H}\mathbf{A}^{-1}\mathbf{s} + \mathbf{n} \\ &= \sqrt{\varrho}\beta\mathbf{H}^H\mathbf{H}\mathbf{t} + \mathbf{n},\end{aligned} \tag{7}$$

where the inversion of $\mathbf{A}$ is converted to finding $\mathbf{t}$ in the linear system of (6) as they yield the same result by $\mathbf{G}\mathbf{s} = \beta\mathbf{H}\mathbf{A}^{-1}\mathbf{s} = \beta\mathbf{H}\mathbf{t}$. As such, the optimization problem based on (7) becomes

$$\min_{\mathbf{t}} E_{\mathbf{n},\mathbf{s}}[\|\mathbf{y} - \mathbf{s}\|^2]. \tag{8}$$

Clearly, after finding $\mathbf{t}$, one can easily get $\beta\mathbf{H}\mathbf{t}$, which is equivalent to obtaining $\mathbf{G}\mathbf{s}$, thus completing the desired precoding operation. Hence, a number of low-complexity TPC algorithms based on the classic iterative methods have been proposed to circumvent the matrix inversion [19], [20], [21], [22], [23], [24], [25].

B. Traditional Low-Complexity Iterative Transmit Precoding

Specifically, given the source information $\mathbf{s}$ and the matrix $\mathbf{A}$, the problem in (8) can be further refined as

$$\min_{\mathbf{t}} \|\mathbf{A}\mathbf{t} - \mathbf{s}\|^2, \tag{9}$$

and the traditional low-complexity iterative methods can be employed by linear TPC to solve it.

According to matrix splitting theory associated with a matrix $\mathbf{Q} \in \mathbb{C}^{K\times K}$ and a nonsingular matrix $\mathbf{P} \in \mathbb{C}^{K\times K}$ obeying $\mathbf{A} = \mathbf{P} + \mathbf{Q}$, the iterations required for solving the linear system in (6) can be expressed in the general form of [47]

$$\mathbf{t}^{k+1} = \mathbf{B}\mathbf{t}^{k} + \mathbf{f}, \tag{10}$$

where k is the iteration index and $\mathbf{B} = -\mathbf{P}^{-1}\mathbf{Q} = \mathbf{I} - \mathbf{P}^{-1}\mathbf{A} \in \mathbb{C}^{K\times K}$ is the *iteration matrix* associated with $\mathbf{f} = \mathbf{P}^{-1}\mathbf{s} \in \mathbb{C}^{K}$. Theoretically, the above iteration converges if [48]

$$\lim_{k\to\infty} \mathbf{B}^{k} = \mathbf{0}, \tag{11}$$

and $\mathbf{t}^{k}$ will gradually converge to the target solution of $\mathbf{t}^{*} = \mathbf{A}^{-1}\mathbf{s}$ after a certain number of iterations. Intuitively, the iterative methods of (10) tend to depend on the selections of $\mathbf{P}$ and $\mathbf{Q}$ in the matrix splitting operation of $\mathbf{A} = \mathbf{P} + \mathbf{Q}$, where the different choices of $\mathbf{P}$ and $\mathbf{Q}$ naturally result in different convergence performance.

In particular, for $\mathbf{P} = \mathbf{D}$ and $\mathbf{Q} = \mathbf{E} + \mathbf{U}$, the iterations of the Jacobi method obey [47]

$$\mathbf{D}\mathbf{t}^{k+1} = -(\mathbf{U} + \mathbf{E})\mathbf{t}^{k} + \mathbf{s}, \tag{12}$$

where the matrices $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{U}$ stand for the diagonal, the lower triangular, and the upper triangular components of the matrix $\mathbf{A}$ associated with $\mathbf{A} = \mathbf{D} + \mathbf{E} + \mathbf{U}$ and $\mathbf{E} = \mathbf{U}^{H}$. To be more specific, the update of each element of $\mathbf{t}$ in (12) can be expressed as

$$t_i^{k+1} = \frac{1}{d_{i,i}}\left(s_i - \sum_{j=1}^{i-1} e_{i,j}t_j^{k} - \sum_{j=i+1}^{K} u_{i,j}t_j^{k}\right), \tag{13}$$

where $u_{i,j} \in \mathbb{R}$, $e_{i,j} \in \mathbb{R}$ and $d_{i,j} \in \mathbb{R}$ standard for the element of the matrices $\mathbf{U}$, $\mathbf{E}$ and $\mathbf{D}$ respectively. Similarly, for the Richardson iteration, $\mathbf{P}$ and $\mathbf{Q}$ are set as $\mathbf{P} = \frac{1}{\omega}\mathbf{I}$ and $\mathbf{Q} = \mathbf{A} - \frac{1}{\omega}\mathbf{I}$ respectively, where $\omega > 0$ is the relaxation factor [29]. However, due to the convergence requirement in (11), the Jacobi iteration works when the matrix $\mathbf{A}$ is strictly diagonally dominant (SDD)[1] while the Richardson iteration is convergent if $0 < \omega < \frac{2}{\lambda_{\max}(\mathbf{A})}$, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of a matrix [45].

To achieve a faster convergence, the Gauss-Seidel (GS) iteration associated with $\mathbf{P} = \mathbf{D} + \mathbf{U}$ and $\mathbf{Q} = \mathbf{E}$ can be used [49]

$$(\mathbf{D} + \mathbf{U})\mathbf{t}^{k+1} = -\mathbf{E}\mathbf{t}^{k} + \mathbf{s}, \tag{14}$$

where we have

$$t_i^{k+1} = \frac{1}{d_{i,i}}\left(s_i - \sum_{j=1}^{i-1} e_{i,j}t_j^{k} - \sum_{j=i+1}^{K} u_{i,j}t_j^{k+1}\right). \tag{15}$$

In sharp contrast to the Jacobi and Richardson iterations and so on, the GS iterations always results in exponential convergence, since the associated iteration matrix $\mathbf{B}_{\text{gs}}$ satisfies $\lambda_{\max}(\mathbf{B}_{\text{gs}}) < 1$ without any extra requirements. This also implies the convergence condition in (11) is always fulfilled, which is also known as *global convergence* [37]. Undoubtedly, global convergence is highly desired for the iterative TPC schemes to make them sufficiently flexible for the various massive MIMO scenarios of interest. As for the computational complexity, it has been shown in [18] that the complexities of these traditional iterative methods are on the order of $\mathcal{O}(K^2k)$, hence considerable complexity reduction can be achieved by linear TPC.

[1]This actually corresponds to $N \gg K$ with respect to the system dimensions of $\mathbf{H}$.

III. Random Block-Based Iterative Transmit Precoding Algorithm

In this section, we commence by discussing the essential difference of these traditional iterative TPC schemes. Then, both ordered and block-based updates are proposed, which then leads to the random block-based iterative transmit precoding (RBI-TPC) concept.

A. Revisiting Traditional Iterative Precoding

On one hand, the Jacobi iteration shown in (13) can be rewritten as

$$t_i^{k+1} = t_i^{k} + \frac{1}{a_{i,i}}\left(s_i - \sum_{j=1}^{i} a_{i,j}t_j^{k} - \sum_{j=i+1}^{K} a_{i,j}t_j^{k}\right), \tag{16}$$

where $\mathbf{H}_{:,i}$ denotes the i-th column of the matrix $\mathbf{H}$ and $a_{i,j} = \mathbf{H}_{:,i}^{H}\mathbf{H}_{:,j}$ is an entry of the matrix $\mathbf{A}$. Clearly, the update of each element in $\mathbf{t}^{k+1}$ of (16) only depends on the results of the previous iteration, which can be further expressed as

$$t_i^{k+1} = t_i^{k} + \frac{1}{a_{i,i}}\left(s_i - \mathbf{A}_{:,i}^{H}\mathbf{t}^{k}\right) \tag{17}$$

with $\mathbf{t}^{k} = [t_1^{k}, \ldots, t_K^{k}]^{H}$. Therefore, given $\mathbf{t}^{k}$, the Jacobi iteration allows t_i^{k+1} to be processed by a parallel structure, which substantially simplifies the hardware implementation in practice.

On the other hand, the GS iteration in (15) can be expressed as

$$t_i^{k+1} = t_i^{k} + \frac{1}{a_{i,i}}\left(s_i - \sum_{j=1}^{i} a_{i,j}t_j^{k} - \sum_{j=i+1}^{K} a_{i,j}t_j^{k+1}\right). \tag{18}$$

In contrast to the Jacobi iteration of (16), in GS iteration the updated elements $[t_{i+1}^{k+1}, \ldots, t_K^{k+1}]$ rather than $[t_{i+1}^{k}, \ldots, t_K^{k}]$ are used for updating t_i^{k+1}, $1 \le i \le K$, which results in an improved convergence performance. In other words, the GS iteration relies on a sequential structure to update t_i^{k+1}, where the latest updates of t_j^{k+1}, $(i+1) < j \le K$ at the current iteration are also taken into account. For a more compact presentation, we reformulate the update in (18) as

$$t_i^{k+1} = t_i^{k} + \frac{1}{a_{i,i}}\left(s_i - \mathbf{A}_{:,i}^{H}\bar{\mathbf{t}}^{k}\right) \tag{19}$$

with $\bar{\mathbf{t}}^{k} = [t_1^{k}, \ldots, t_i^{k}, t_{i+1}^{k+1}, \ldots, t_K^{k+1}]^{H}$. Note that the update of t_i^{k+1} in the GS iteration of (15), (18), and (19) are illustrated in a backward oriented order from $i = K$ to $i = 1$, where the forward oriented order from $i = 1$ to $i = K$ is straightforward based on commencing the GS iteration with respect to

$$(\mathbf{D} + \mathbf{E})\mathbf{t}^{k+1} = -\mathbf{U}\mathbf{t}^k + \mathbf{s}. \quad (20)$$

B. Algorithm Description

Different from Jacobi iteration that allows the parallel updating t_i^{k+1} at a time, we can see from the above analysis that the update of t_i^{k+1} in the GS iteration partially depends on the most recently updated results, which is essentially determined by the updating order of the iterations. For example, t_i^{k+1} is partially decided by $\{t_{i+1}^{k+1}, \ldots, t_K^{k+1}\}$ in the backward oriented updating order or by $\{t_1^{k+1}, \ldots, t_{i-1}^{k+1}\}$ in the forward oriented updating order. However, the backward or forward oriented updating order is only used for the ease of exposition. But the natural question arising is, *what is the best order in the GS iteration*?

To answer this question, we commence on the investigation by firstly reexamining the GS iteration relying on a random updating order. To be more specific, given the iteration in (19), the coordinate i is randomly sampled from a certain distribution using the sampling probability p_i, which is formulated as

$$t_i^{k+1} = t_i^k + \frac{1}{a_{i,i}}\left(s_i - \mathbf{A}_{:,i}^H \bar{\mathbf{t}}^k\right) \text{ with } i \sim p_i, \quad (21)$$

where $\Sigma_{i=1}^K p_i = 1$.

Another observation concerning the traditional iterative TPC schemes is that all of them only update a single component of $\mathbf{t}$ at each instant, which can be clearly seen in (17), (19) and so on. Therefore, another natural question arises: *Is it possible to update multiple components of* $\mathbf{t}$ *instead of a single component of* $\mathbf{t}$ *at each instant*? If so, the correlation among the components of $\mathbf{t}$ can be beneficially exploited for expediting the convergence. Then, the original single-component update of $\mathbf{t}$ can be viewed as a special case of the multi-component update, and we term this multi-component update as *block-based update*.

Therefore, motivated by the concept of block-based update, we further upgrade the iteration in (21) as follows

$$\mathbf{t}_{\mathcal{Q}_j}^{k+1} = \mathbf{t}_{\mathcal{Q}_j}^k + (\mathbf{H}_{:,\mathcal{Q}_j}^H \mathbf{H}_{:,\mathcal{Q}_j})^{-1}\left(\mathbf{s}_{\mathcal{Q}_j} - \mathbf{A}_{:,\mathcal{Q}_j}^H \bar{\mathbf{t}}^k\right) \quad (22)$$

with coordinates index set (i.e., block index) $\mathcal{Q}_j$ obeying the sampling probability $p_{\mathcal{Q}_j}$ as

$$\mathcal{Q}_j \sim p_{\mathcal{Q}_j} \text{ and } \Sigma_{\mathcal{Q}_j} p_{\mathcal{Q}_j} = 1, \quad (23)$$

which leads to the proposed random block-based iterative transmit precoding (RBI-TPC) algorithm. Here, the set $\mathcal{Q}_j$, $1 \le j \le r$, contains multiple (i.e., $|\mathcal{Q}_j| = q_j$) coordinates from the set $\{1, \ldots, K\}$, e.g.,

$$\underbrace{\{1,3,5\}}_{\mathcal{Q}_1} \cup \ldots \cup \underbrace{\{2,6,8\}}_{\mathcal{Q}_r} = \{1, \ldots, K\} \quad (24)$$

with $\mathcal{Q}_a \cap \mathcal{Q}_b = \emptyset$, $1 \le a \ne b \le r$. In coordinate index set $\mathcal{Q}_j$, $\mathbf{I}_{:,\mathcal{Q}_j}$ is a column-based partition of identity matrix $\mathbf{I}$. For example, for a 3×3 identity matrix $\mathbf{I}$ with $\mathcal{Q}_j = \{1, 3\}$, we have

$$\mathbf{I}_{:,\mathcal{Q}_j} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (25)$$

Accordingly, given $\mathbf{I}_{:,\mathcal{Q}_j}$, we have $\mathbf{H}_{:,\mathcal{Q}_j} = \mathbf{H}\mathbf{I}_{:,\mathcal{Q}_j} \in \mathbb{C}^{N \times q_j}$, $\mathbf{A}_{:,\mathcal{Q}_j} = \mathbf{A}\mathbf{I}_{:,\mathcal{Q}_j} \in \mathbb{C}^{K \times q_j}$, $\mathbf{t}_{\mathcal{Q}_j} = \mathbf{I}_{:,\mathcal{Q}_j}^H \mathbf{t} \in \mathbb{C}^{q_j}$ and $\mathbf{s}_{\mathcal{Q}_j} = \mathbf{I}_{:,\mathcal{Q}_j}^H \mathbf{s} \in \mathbb{C}^{q_j}$. Note that the coordinates in each set $\mathcal{Q}_j$ are fixed initially for simplicity. Furthermore, for the sake of notational simplicity, the same block size of $q_1 = \ldots = q_r = K/r = q$ is applied. Intuitively, given block size of $q = 1$, the block-based iteration in (22) reduces to the standard one as in (21).

To facilitate the following analysis, we rewrite the iteration equation of (22) as

$$\begin{aligned}\mathbf{t}^{l+1} &= \mathbf{t}^l + \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}_{:,\mathcal{Q}_j}^H \mathbf{H}_{:,\mathcal{Q}_j})^{-1}\left(\mathbf{s}_{\mathcal{Q}_j} - \mathbf{A}_{:,\mathcal{Q}_j}^H \mathbf{t}^l\right) \\ &= \mathbf{t}^l + \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}_{:,\mathcal{Q}_j}^H \mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}_{:,\mathcal{Q}_j}^H\left(\mathbf{s} - \mathbf{A}\mathbf{t}^l\right) \\ &= \mathbf{t}^l + \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{I}_{:,\mathcal{Q}_j}^H \mathbf{A}\mathbf{I}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}_{:,\mathcal{Q}_j}^H\left(\mathbf{s} - \mathbf{A}\mathbf{t}^l\right) \quad (26)\end{aligned}$$

with $\mathcal{Q}_j \sim p_{\mathcal{Q}_j}$, where q elements of the vector $\mathbf{t} \in \mathbb{C}^K$ are updated at each time. Here we use $l \ge 0$ to denote the iteration index and $\mathbf{t}^l = [t_1^l, \ldots, t_K^l]^H$. As for the initial setup $\mathbf{t}^0$, we can use an arbitrary point and here we set $\mathbf{t}^0 = \mathbf{D}^{-1}\mathbf{s}$ by default. The proposed RBI-TPC algorithm constructed for the downlink of massive MIMO is outlined in Algorithm 1 at a glance. In fact, the traditional GS iteration can be viewed as a coordinate descent algorithm since it only updates a single component at a time [50]. Therefore, as an extension of GS iterations with the updated processing order and block operations, the proposed RBI-TPC can be essentially viewed as a variant of the randomized block coordinate descent algorithm [51].

Since the traditional GS iteration updates all the K components of $\mathbf{t}$ in a single iteration indexed by k (i.e., performs K single component updates at each iteration), for a fair comparison we define a *full iteration* for the proposed RBI-TPC algorithm, which contains K/q iterations in (26). In this way, the full iteration of RBI-TPC can be calibrated by the index k. As for the associated complexity, thanks to the special structure of $\mathbf{I}_{:,\mathcal{Q}_j}$, the calculation of $(\mathbf{H}_{:,\mathcal{Q}_j}^H \mathbf{H}_{:,\mathcal{Q}_j})^{-1}$ (i.e., $(\mathbf{I}_{:,\mathcal{Q}_j}^H \mathbf{A}\mathbf{I}_{:,\mathcal{Q}_j})^{-1}$) in (22) can be realized at a computational complexity order of $O(q^3)$. As for the complexity of $(\mathbf{s}_{\mathcal{Q}_j} - \mathbf{A}_{:,\mathcal{Q}_j}^H \bar{\mathbf{t}}^k)$, it is $O(Kq)$, while multiplying it with $(\mathbf{H}_{:,\mathcal{Q}_j}^H \mathbf{H}_{:,\mathcal{Q}_j})^{-1}$ is on the order of $O(q^2)$. Therefore, the overall complexity of computing (22) in terms of the number of complex multiplications can be expressed by $O(q^3 + Kq)$. Then, based on these, the complexity of a full iteration in the proposed RBI-TPC algorithm turns out to be $O(q^2 K + K^2)$. As it transpires, given the block size $1 \le q \le \sqrt{K}$, the complexity order of RBI-TPC is $O(K^2)$, which is competitive compared to the traditional iterative TPC schemes. For example, the full iteration complexity of the modified randomized iterative precoding algorithm (MRIPA) in [37] is $O(q^2 K + 4K^2)$. Although it is still expressed by $O(K^2)$ for $1 \le q \le \sqrt{K}$, the complexity of MRIPA is higher than that of RBI-TPC. Moreover, we will demonstrate that considerable convergence acceleration can be obtained by the block-based update and the reasonable choice of the sampling distribution $p_{\mathcal{Q}_j}$ in the following.

Algorithm 1 Random Block-Based Iterative Transmit Precoding (RBI-TPC) Algorithm

Input: $\mathbf{A} = \mathbf{H}^H\mathbf{H} + \xi\mathbf{I}$, L, β, q

Output: Approximated RZF precoding solution $\mathbf{Gs} = \beta\mathbf{Ht}^L$

1: **for** $l = 0, \ldots, L-1$ **do**

2: get the block set $\mathcal{Q}_j$ by random sampling according to $p_{\mathcal{Q}_j}$

3: update $\mathbf{t}^{l+1}$ based on (26)

4: **end for**

5: output $\mathbf{Gs} = \beta\mathbf{Ht}^L$

IV. Convergence Analysis and Optimization

In this section, the convergence behaviour of the proposed RBI-TPC is analyzed in full detail. The globally exponential convergence of RBI-TPC is demonstrated by relying on the derived convergence rate, followed by the discussion of determining the sampling distribution $p_{\mathcal{Q}_j}$.

A. Globally Exponential Convergence

To start with, let $\mathbf{t}^* = \mathbf{A}^{-1}\mathbf{s}$ denote the solution of (6). Then we can get the following result.

Lemma 1: In the proposed RBI-TPC algorithm, the vector $\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^*)$ *is perpendicular to the vector* $\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^l)$ *as*

$$\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^*) \perp \mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^l). \tag{27}$$

Proof: First of all, for notational simplicity, we define

$$\mathbf{r}_l = \mathbf{t}^l - \mathbf{t}^*. \tag{28}$$

Then, to prove the relationship in (27), one has to show that

$$(\mathbf{t}^{l+1} - \mathbf{t}^l)^H\mathbf{H}^H\mathbf{H}\mathbf{r}_{l+1} = \mathbf{0}, \tag{29}$$

which is demonstrated in the following

$$\begin{aligned}
&(\mathbf{t}^{l+1} - \mathbf{t}^l)^H\mathbf{H}^H\mathbf{H}\mathbf{r}_{l+1} \\
&= (\mathbf{t}^{l+1} - \mathbf{t}^l)^H\mathbf{A}\mathbf{r}_{l+1} \\
&= (\mathbf{t}^{l+1} - \mathbf{t}^l)^H\mathbf{A}(\mathbf{r}_l - \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l) \\
&= [\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}^H_{:,\mathcal{Q}_j}(\mathbf{s} - \mathbf{A}\mathbf{t}^l)]^H \\
&\quad \times \mathbf{A}[\mathbf{r}_l - \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l] \\
&= (\mathbf{s} - \mathbf{A}\mathbf{t}^l)^H\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-H}\mathbf{I}^H_{:,\mathcal{Q}_j} \\
&\quad \times \mathbf{A}[\mathbf{r}_l - \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l] \\
&= (\mathbf{s} - \mathbf{A}\mathbf{t}^l)^H\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-H} \\
&\quad \times [\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l - \mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l] \\
&= (\mathbf{s} - \mathbf{A}\mathbf{t}^l)^H\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-H}(\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l - \mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{A}\mathbf{r}_l) \\
&= \mathbf{0}.
\end{aligned} \tag{30}$$

■

Based on Lemma 1, one can readily show that

$$\|\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^*)\|^2 = \|\mathbf{H}(\mathbf{t}^l - \mathbf{t}^*)\|^2 - \|\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^l)\|^2 \tag{31}$$

using the classic *Pythagorean theorem*. Moreover, according to (31), the following result can be inferred to show the globally exponential convergence of the proposed RBI-TPC in terms of the mean squared error (MSE).

Theorem 1: For the downlink of massive MIMO systems, the proposed RBI-TPC converges by

$$E[\|\mathbf{H}(\mathbf{t}^l - \mathbf{t}^*)\|^2] \le \rho^l\|\mathbf{H}(\mathbf{t}^0 - \mathbf{t}^*)\|^2 \tag{32}$$

with the global convergence rate of

$$\rho = 1 - \lambda_{\min}(E[\mathbf{Z}]) < 1. \tag{33}$$

Proof: First of all, according to (31), given $\mathbf{t}^l$, it follows that

$$E[\|\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^*)\|^2] = \|\mathbf{H}(\mathbf{t}^l - \mathbf{t}^*)\|^2 - E[\|\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^l)\|^2]. \tag{34}$$

Then, let us focus on the term $E[\|\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^l)\|^2]$ shown above, yielding

$$\begin{aligned}
E[\|\mathbf{H}(\mathbf{t}^{l+1} - \mathbf{t}^l)\|^2] &= E[\|\mathbf{H}\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}^H_{:,\mathcal{Q}_j}\mathbf{H}^H\mathbf{H}\mathbf{r}_l\|^2] \\
&\overset{(a)}{=} E[\|\mathbf{Z}\mathbf{H}\mathbf{r}_l\|^2] \\
&= E[\mathbf{r}_l^H\mathbf{H}^H\mathbf{Z}^H\mathbf{Z}\mathbf{H}\mathbf{r}_l] \\
&\overset{(b)}{=} E[\mathbf{r}_l^H\mathbf{H}^H\mathbf{Z}^H\mathbf{H}\mathbf{r}_l] \\
&= \mathbf{r}_l^H\mathbf{H}^H E[\mathbf{Z}]\mathbf{H}\mathbf{r}_l,
\end{aligned} \tag{35}$$

where the following definition of the symmetric matrix $\mathbf{Z} \in \mathbb{C}^{N\times N}$ is applied in (a)

$$\mathbf{Z} \triangleq \mathbf{H}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{H}^H_{:,\mathcal{Q}_j} = \mathbf{Z}^H \tag{36}$$

and equality (b) holds due to

$$\begin{aligned}
\mathbf{Z}^H\mathbf{Z} &= \mathbf{H}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{H}^H_{:,\mathcal{Q}_j} \\
&= \mathbf{H}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{H}^H_{:,\mathcal{Q}_j} \\
&= \mathbf{Z}.
\end{aligned} \tag{37}$$

More specifically, the expectation of matrix $\mathbf{Z}$ is symmetric positive definite due to its special structure

$$\begin{aligned}
&E[\mathbf{Z}] \\
&= \sum_{j=1}^{r} p_{\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{H}^H_{:,\mathcal{Q}_j} \\
&= \mathbf{H}\left(\sum_{j=1}^{r} p^{\frac{1}{2}}_{\mathcal{Q}_j}\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-\frac{1}{2}}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-\frac{1}{2}}\mathbf{I}^H_{:,\mathcal{Q}_j}p^{\frac{1}{2}}_{\mathcal{Q}_j}\right)\mathbf{H}^H \\
&= (\mathbf{H}\mathbf{J})(\mathbf{J}^H\mathbf{H}^H) \\
&= \mathbf{V}\mathbf{V}^H,
\end{aligned} \tag{38}$$

where $\mathbf{V} = \mathbf{H}\mathbf{J} \in \mathbb{C}^{N\times K}$ and the matrix $\mathbf{J} \in \mathbb{C}^{K\times K}$ is invertible blocked diagonal as

$$\mathbf{J} = \mathrm{diag}(p^{\frac{1}{2}}_{\mathcal{Q}_1}(\mathbf{H}^H_{:,\mathcal{Q}_1}\mathbf{H}_{:,\mathcal{Q}_1})^{-\frac{1}{2}}, \ldots, p^{\frac{1}{2}}_{\mathcal{Q}_r}(\mathbf{H}^H_{:,\mathcal{Q}_r}\mathbf{H}_{:,\mathcal{Q}_r})^{-\frac{1}{2}}). \tag{39}$$

Based on (35) and (38), we have

$$
\begin{aligned}
E[\|\mathbf{H}(\mathbf{t}^{l+1}-\mathbf{t}^{l})\|^2] &= \mathbf{r}_l^H\mathbf{H}^H\mathbf{V}\mathbf{V}^H\mathbf{H}\mathbf{r}_l \\
&= \|\mathbf{V}^H\mathbf{H}\mathbf{r}_l\|^2 \\
&\overset{(c)}{\geq} \lambda_{\min}(\mathbf{V}^H\mathbf{V})\|\mathbf{H}\mathbf{r}_l\|^2 \\
&\overset{(d)}{=} \lambda_{\min}(E[\mathbf{Z}])\|\mathbf{H}(\mathbf{t}^{l}-\mathbf{t}^{*})\|^2. \quad (40)
\end{aligned}
$$

Here, the inequality in (c) holds due to the fact that

$$\|\mathbf{B}^H\mathbf{x}\|^2 \geq \lambda_{\min}(\mathbf{B}^H\mathbf{B})\|\mathbf{x}\|^2 \quad (41)$$

for any matrix $\mathbf{B}$ and vector $\mathbf{x}$, and (d) follows

$$\lambda_{\min}(\mathbf{V}^H\mathbf{V}) = \lambda_{\min}(\mathbf{V}\mathbf{V}^H) = \lambda_{\min}(E[\mathbf{Z}]). \quad (42)$$

Next, according to (34) and (40), we can arrive at the following result

$$E[\|\mathbf{H}(\mathbf{t}^{l+1}-\mathbf{t}^{*})\|^2] \leq (1-\lambda_{\min}(E[\mathbf{Z}]))\|\mathbf{H}(\mathbf{t}^{l}-\mathbf{t}^{*})\|^2. \quad (43)$$

Meanwhile, since $E[\mathbf{Z}]$ is positive definite, all its eigenvalues are larger than 0 so that

$$\rho = 1-\lambda_{\min}(E[\mathbf{Z}]) < 1. \quad (44)$$

Consequently, it follows that

$$
\begin{aligned}
E[\|\mathbf{H}(\mathbf{t}^{l+1}-\mathbf{t}^{*})\|^2] &\leq \rho\|\mathbf{H}(\mathbf{t}^{l}-\mathbf{t}^{*})\|^2 \\
&\leq \rho^2\|\mathbf{H}(\mathbf{t}^{l-1}-\mathbf{t}^{*})\|^2 \\
&\leq \cdots \\
&\leq \rho^{l+1}\|\mathbf{H}(\mathbf{t}^{0}-\mathbf{t}^{*})\|^2, \quad (45)
\end{aligned}
$$

which completes the proof. ∎

B. The Choices of Sampling Distribution $p_{\mathcal{Q}_j}$

According to Theorem 1, the convergence of the RBI-TPC algorithm chiefly depends on the expectation of the matrix $\mathbf{Z}$, which is partially determined by the sampling probability $p_{\mathcal{Q}_j}$. Therefore, we aim for specifying the convergence rate given different choices of sampling probabilities.

Intuitively, a natural choice for $p_{\mathcal{Q}_j}$ is uniform distribution, which leads to the following result.

Corollary 1: With $p_{\mathcal{Q}_j}$ following the uniform sampling probability

$$p_{\mathcal{Q}_j} = \frac{1}{r}, \quad (46)$$

the proposed RBI-TPC converges by

$$E[\|\mathbf{H}(\mathbf{t}^{l}-\mathbf{t}^{*})\|^2] \leq \rho_{\text{uniform}}^l\|\mathbf{H}(\mathbf{t}^{0}-\mathbf{t}^{*})\|^2 \quad (47)$$

with

$$\rho_{\text{uniform}} = 1-\frac{1}{r}\cdot\frac{\lambda_{\min}(\mathbf{H}^H\mathbf{H})}{\lambda_{\max}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}. \quad (48)$$

Proof: Specifically, according to (38), it follows that

$$
\begin{aligned}
\lambda_{\min}(E[\mathbf{Z}]) &= \lambda_{\min}(E[\mathbf{H}\mathbf{J}\mathbf{J}^H\mathbf{H}^H]) \\
&\overset{(e)}{\geq} \lambda_{\min}(E[\mathbf{H}^H\mathbf{H}])\lambda_{\min}(\mathbf{J}^2) \\
&= \lambda_{\min}(\mathbf{H}^H\mathbf{H})\lambda_{\min}(\mathbf{J}^2) \quad (49) \\
&= \frac{1}{r}\cdot\frac{\lambda_{\min}(\mathbf{H}^H\mathbf{H})}{\lambda_{\max}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}, \quad (50)
\end{aligned}
$$

where the inequality (e) comes from $\lambda_{\min}(\mathbf{E}\mathbf{F}) \geq \lambda_{\min}(\mathbf{E})\lambda_{\min}(\mathbf{F})$ for any positive definite matrices $\mathbf{E}$ and $\mathbf{F}$. Therefore, this completes the proof by simply substituting (50) into (44). ∎

From (48), it seems that the convergence rate ρ_{uniform} is also related to the matrix partition of $\mathbf{H}$ (i.e., $\mathbf{H}_{:,\mathcal{Q}_j}$). However, a partition $\mathbf{H}_{:,\mathcal{Q}_j}$ designed according to a certain criterion seems unnecessary, since each block $\mathbf{H}_{:,\mathcal{Q}_j}$ is sampled uniformly based on (46). In addition to the uniform sampling probability, its counterpart based on the matrix Frobenius norm is also studied below.

Corollary 2: With $p_{\mathcal{Q}_j}$ following the sampling probability

$$p_{\mathcal{Q}_j} = \frac{\|\mathbf{H}_{:,\mathcal{Q}_j}\|_F^2}{\|\mathbf{H}\|_F^2}, \quad (51)$$

the proposed RBI-TPC converges by

$$E[\|\mathbf{H}(\mathbf{t}^{l}-\mathbf{t}^{*})\|^2] \leq \rho_{\text{norm}}^l\|\mathbf{H}(\mathbf{t}^{0}-\mathbf{t}^{*})\|^2 \quad (52)$$

with

$$\rho_{\text{norm}} = 1-\alpha\cdot\frac{\lambda_{\min}(\mathbf{H}^H\mathbf{H})}{\text{Tr}(\mathbf{H}^H\mathbf{H})}, \quad (53)$$

where

$$\alpha \triangleq \min_j\left\{\frac{\text{Tr}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}{\lambda_{\max}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}\right\} \geq 1. \quad (54)$$

Proof: Due to $\|\mathbf{B}\|_F^2 = \sum_i\sum_j|b_{i,j}|^2 = \text{Tr}(\mathbf{B}^H\mathbf{B})$, with the sampling probability $p_{\mathcal{Q}_j}$ in (51), it follows that

$$
\begin{aligned}
\lambda_{\min}(\mathbf{J}^2) &= \frac{1}{\text{Tr}(\mathbf{H}^H\mathbf{H})}\min_j\left\{\frac{\text{Tr}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}{\lambda_{\max}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}\right\} \\
&= \frac{\alpha}{\text{Tr}(\mathbf{H}^H\mathbf{H})}, \quad (55)
\end{aligned}
$$

with $\alpha \geq 1$ due to

$$\text{Tr}(\mathbf{B}) = \sum_i\lambda_i(\mathbf{B}) \geq \lambda_{\max}(\mathbf{B}) \geq \lambda_{\min}(\mathbf{B}) \quad (56)$$

for any square matrix $\mathbf{B}$.

Then, based on (49) and (55), the convergence rate ρ in (44) becomes

$$\rho_{\text{norm}} = 1-\alpha\cdot\frac{\lambda_{\min}(\mathbf{H}^H\mathbf{H})}{\text{Tr}(\mathbf{H}^H\mathbf{H})}, \quad (57)$$

completing the proof. ∎

According to α in (53), given the block size $q = K/r = |\mathcal{Q}_j|$, the convergence rate ρ_{norm} depends on the specific choices of the matrix partition $\mathbf{H}_{:,\mathcal{Q}_j}$, where finding the optimal solution turns out to be computationally expensive. For the sake of efficiency, here we propose to build the partition $\mathbf{H}_{:,\mathcal{Q}_j}$ according to the descending order of $\|\mathbf{H}_{:,i}\|$ (i.e., $\|\mathbf{h}_i\|$), namely,

$$\|\mathbf{H}_{:,i_1}\| \geq \|\mathbf{H}_{:,i_2}\| \geq \cdots \geq \|\mathbf{H}_{:,i_K}\| \quad (58)$$

with

$$\underbrace{\{i_1, i_2, \ldots\}}_{\mathcal{Q}_1} \cup \ldots \cup \underbrace{\{\ldots, i_K\}}_{\mathcal{Q}_r} = \{1, \ldots, K\} \quad (59)$$

Clearly, in this way, we can see that

$$\|\mathbf{H}_{:,\mathcal{Q}_1}\|_F \geq \cdots \geq \|\mathbf{H}_{:,\mathcal{Q}_r}\|_F. \tag{60}$$

Here, we point out that the calculations of the Frobenius norm of $\mathbf{H}_{:,j}$ are only performed once at the beginning, which can be readily carried out during the preprocessing stage. Apart from the matrix partition $\mathbf{H}_{:,\mathcal{Q}_j}$, as shown in (49), the condition number of the matrix $\mathbf{H}^H\mathbf{H}$ (i.e., $\kappa = \lambda_{\text{max}}(\mathbf{H}^H\mathbf{H})/\lambda_{\text{min}}(\mathbf{H}^H\mathbf{H})$) also plays an important role in determining the convergence rate ρ_{norm}, where a smaller κ - which also corresponds to a more orthogonal matrix $\mathbf{H}$) - is preferable for attaining more rapid convergence.

V. Ordered Block-Based Iterative Transmit Precoding Algorithm

In this section, by adopting the concept of multi-step conditional sampling, we extend the RBI-TPC to the proposed ordered block-based iterative transmit precoding (OBI-TPC) algorithm, where remarkable gains in the both convergence and efficiency can be attained.

A. Extension by Multi-Step Conditional Sampling

In [37], the concept of multi-step conditional sampling is applied for replacing the standard sampling for the sake of the improved iteration convergence and efficiency. Inspired by this, we now adopt the multi-step conditional sampling to the proposed RBI-TPC to achieve improved gains. Specifically, for the sake of notational simplicity, let $\mathcal{Q}^l$ denote the choice of the set $\mathcal{Q}$ at iteration l. Then we define the multi-step conditional sampling probability in RBI-TPC as

$$\overline{p}^f_{\mathcal{Q}_j} \triangleq p(\mathcal{Q}^l = \mathcal{Q}_j | \mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}) \tag{61}$$

$$= \frac{p(\mathcal{Q}_j)}{1 - p(\mathcal{Q}^{l-1}) - \cdots - p(\mathcal{Q}^{l-f})} \tag{62}$$

with $\mathcal{Q}_j \notin \{\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}\}$, where $0 \leq f \leq (r-1)$ denotes the length of the multi-step conditional sampling. By doing this, we can see that the previous sampling results are taken into account in sampling the current set choice of $\mathcal{Q}^l$, which leads to a reduced state space of $\mathcal{Q}$. In other words, the sampling probability $p_{\mathcal{Q}_j}$ may be viewed as a special case of $\overline{p}^f_{\mathcal{Q}_j}$ associated with $f = 0$.

Based on the multi-step conditional sampling probability in (61), we can see that the globally exponential convergence of RBI-TPC still holds.

Theorem 2: For the downlink of massive MIMO systems, let the set $\mathcal{Q}_j$ be sampled from the multi-step conditional sampling probability $\overline{p}^f_{\mathcal{Q}_j}$ in (61), the proposed RBI-TPC converges by

$$E[\|\mathbf{H}(\mathbf{t}^l - \mathbf{t}^*)\|^2] \leq \overline{\rho}^f \|\mathbf{H}(\mathbf{t}^{l-1} - \mathbf{t}^*)\|^2 \tag{63}$$

with the global convergence rate

$$\overline{\rho}^f = 1 - \lambda_{\text{min}}(E[\mathbf{Z}|\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}]) < 1. \tag{64}$$

Proof: The proof is similar to that of Theorem 1 but relies on the conditional expectation of the matrix $\mathbf{Z}$, which is formulated as

$$\begin{aligned}
&E[\mathbf{Z}|\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}] \\
&= \sum_{j,\mathcal{Q}_j \notin \mathcal{Q}^{l-1},\ldots,\mathcal{Q}^{l-f}}^{r} \overline{p}^f_{\mathcal{Q}_j} \mathbf{H}_{:,\mathcal{Q}_j} (\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-1} \mathbf{H}^H_{:,\mathcal{Q}_j} \\
&= \sum_{j,\mathcal{Q}_j \notin \mathcal{Q}^{l-1},\ldots,\mathcal{Q}^{l-f}}^{r} \Big((\overline{p}^f_{\mathcal{Q}_j})^{\frac{1}{2}} \mathbf{H}_{:,\mathcal{Q}_j} (\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-\frac{1}{2}} \\
&\quad \times (\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j})^{-\frac{1}{2}} \mathbf{H}^H_{:,\mathcal{Q}_j} (\overline{p}^f_{\mathcal{Q}_j})^{\frac{1}{2}} \Big) \\
&= (\mathbf{H}\overline{\mathbf{I}}\overline{\mathbf{J}})(\overline{\mathbf{J}}^H\overline{\mathbf{I}}^H\mathbf{H}^H).
\end{aligned} \tag{65}$$

Here, the matrix $\overline{\mathbf{I}} \in \mathbb{C}^{K\times(K-fq)}$ is a partition of the identity matrix $\mathbf{I}$ created by removing the corresponding columns in the set $\{\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}\}$. Similarly, matrix $\overline{\mathbf{J}} \in \mathbb{C}^{(K-fq)\times(K-fq)}$ takes the form of $\overline{\mathbf{J}} = \text{diag}(\overline{p}^{\frac{1}{2}}_{\mathcal{Q}_1}(\mathbf{H}^H_{:,\mathcal{Q}_1}\mathbf{H}_{:,\mathcal{Q}_1})^{-\frac{1}{2}}, \ldots, \overline{p}^{\frac{1}{2}}_{\mathcal{Q}_r}(\mathbf{H}^H_{:,\mathcal{Q}_r}\ \mathbf{H}_{:,\mathcal{Q}_r})^{-\frac{1}{2}})$, where the related terms $\overline{p}^{\frac{1}{2}}_{\mathcal{Q}_i}(\mathbf{H}^H_{:,\mathcal{Q}_i}\mathbf{H}_{:,\mathcal{Q}_i})^{-\frac{1}{2}}$ associated with $\mathcal{Q}_i \in \{\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}\}$ are removed from the matrix $\mathbf{J}$.

On the other hand, it is clear that $E[\mathbf{Z}|\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}]$ is still symmetric positive definite, hence we have $\lambda_{\text{min}}(E[\mathbf{Z}|\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}]) > 0$, which results in the globally exponential convergence performance shown in (64). ■

Based on Theorem 2, we can get the following result to confirm the convergence gain attained by the multi-step conditional sampling.

Corollary 3: With the increment of $0 \leq f \leq (r-1)$, the proposed RBI-TPC with multi-step conditional sampling probability $\overline{p}^f_{\mathcal{Q}_j}$ in (61) achieves a faster convergence performance by a smaller $\overline{\rho}^f$ in (64).

Proof: According to (49), we have

$$\begin{aligned}
&\lambda_{\text{min}}(E[\mathbf{Z}|\mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f}]) \geq \lambda_{\text{min}}(\mathbf{H}^H\mathbf{H})\lambda_{\text{min}}(\overline{\mathbf{J}}^2) \\
&= \min_j \left\{ \overline{p}^f_{\mathcal{Q}_j} \cdot \frac{\lambda_{\text{min}}(\mathbf{H}^H\mathbf{H})}{\lambda_{\text{max}}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j}|\mathcal{Q}_j \notin \mathcal{Q}^{l-1}, \ldots, \mathcal{Q}^{l-f})} \right\}.
\end{aligned} \tag{66}$$

Clearly, the term in (66) increases monotonically with the increment of f due to the monotonicity of both

$$\overline{p}^f_{\mathcal{Q}_j} \geq \overline{p}^{f-1}_{\mathcal{Q}_j} \tag{67}$$

and $\lambda_{\text{max}}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j}|\mathcal{Q}_j \notin \mathcal{Q}^{l-1}) \geq \lambda_{\text{max}}(\mathbf{H}^H_{:,\mathcal{Q}_j}\mathbf{H}_{:,\mathcal{Q}_j}\ |\mathcal{Q}_j \notin \mathcal{Q}^{l-1}, \mathcal{Q}^{l-2})$. Consequently, based on (64) and (66), a smaller $\overline{\rho}^f$ can be obtained, which corresponds to a faster convergence performance. ■

In theory, for iterative TPC schemes, the convergence accuracy is directly coupled with the number of iterations. For example, to achieve the same convergence accuracy, a faster converging solution requires fewer iterations, which results in a lower complexity. Therefore, although the proposed TPC schemes have the same complexity order as the traditional ones at each iteration, our solutions require fewer iterations due to their faster convergence, which leads to a lower complexity at specific target performance.

B. Ordered Block-Based Iterative Transmit Precoding Algorithm

It is plausible from Corollary 3 that the fastest convergence is achieved for $f = (r-1)$. More interestingly, when $f = (r-1)$ and $l > (r-1)$, only a single sampling choice is left for each iteration of RBI-TPC. In this condition, the random iteration in RBI-TPC turns out to be deterministic, yielding the convergence rate of

$$\overline{\rho}_{\mathcal{Q}_j}^{r-1} \leq 1 - \frac{\lambda_{\min}(\mathbf{H}^H\mathbf{H})}{\lambda_{\max}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})}. \quad (68)$$

Intuitively, having a deterministic version of RBI-TPC is highly desired in practice, provided that the process of random sampling can be avoided without performance loss. This not only leads to the convergence enhancement but also results in the efficiency improvement.

Although the proposed RBI-TPC with $f = (r-1)$ step conditional sampling becomes deterministic when the iteration index exceeds $l > (r-1)$, the updating order of the components in $\mathbf{t}$ is actually determined by the sampling probability $p_{\mathcal{Q}_j}$. More specifically, according to the sampling probability in (51), the multiple components of $\mathbf{t}$ associated with the largest $\|\mathbf{H}_{:,\mathcal{Q}_j}\|_F$ are the most likely to be updated at the first iteration. Then, since the choice in the last iteration is removed from the sampling list, the multiple components with the second largest $\|\mathbf{H}_{:,\mathcal{Q}_j}\|_F$ are more likely to be updated at the second iteration and so on.

Therefore, based on the RBI-TPC using $f = (r-1)$ step conditional sampling, inspired by the sampling probability in (51), we propose the ordered block-based iterative transmit precoding (OBI-TPC) algorithm, which updates the components of $\mathbf{t}$ by (26), but in a descending order according to $\|\mathbf{H}_{:,\mathcal{Q}_j}\|_F^2/\|\mathbf{H}\|_F^2$ in (51). In fact, this corresponds to a descending order of $\|\mathbf{H}_{:,\mathcal{Q}_j}\|_F$ shown in (60). To make it more specific, the iteration in the proposed OBI-TPC algorithm proceeds as follows

$$\mathbf{t}^{l+1} = \mathbf{t}^l + \mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}_{:,\mathcal{Q}_j}^H\left(\mathbf{s} - \mathbf{A}\mathbf{t}^l\right) \quad (69)$$

with the descending order and the matrix partition shown in (60) and (59), respectively. In addition to the enhanced convergence speed, based on Theorem 2 and Corollary 3, the proposed OBI-TPC algorithm also enjoys global convergence, which is important for its potential applications in 5G NR. In a nutshell, the operations of OBI-TPC are outlined in Algorithm 2.

Algorithm 2 Ordered Block-Based Iterative Transmit Precoding (OBI-TPC) Algorithm

---

Input: $\mathbf{A} = \mathbf{H}^H\mathbf{H} + \xi\mathbf{I}$, L, β, q
Output: Approximated RZF precoding solution $\mathbf{G}\mathbf{s} = \beta\mathbf{H}\mathbf{t}^L$
1: set the matrix partition according to (59)
2: **for** $l = 0, \ldots, L-1$ **do**
3: update $\mathbf{t}^{l+1}$ according to (26) with the order in (60)
4: **end for**
5: output $\mathbf{G}\mathbf{s} = \beta\mathbf{H}\mathbf{t}^L$

Remark 1: The proposed OBI-TPC algorithm of (69) is an approximation of the RBI-TPC algorithm using $f = (r-1)$ step conditional sampling probability $p_{\mathcal{Q}_j}$ in (51).

On the other hand, since the traditional GS iteration operates either in a forward or backward oriented order, it can be viewed as a deterministic version of RBI-TPC (with block size $q = 1$ and multiple-step $f = (r-1)$) following the random sampling probability in (46). When the iteration index goes $l > (r-1)$, the updating order of t_i becomes fixed without ordering, which is essentially the same as the default forward or backward oriented order.

Remark 2: The traditional GS TPC in (19) can be approximated by the RBI-TPC algorithm with block size $q = 1$ and $f = (r-1)$ step conditional sampling probability $p_{\mathcal{Q}_j}$ in (46).

To make it more specific for a better understanding, Fig. 1 is given to show our convergence comparisons for the proposed RBI-TPC associated with different sampling distributions in a 256×64 massive MIMO system. In particular, RBI-TPC using the uniform sampling distribution $p_{\mathcal{Q}_j}$ in (46) as well as the sampling distribution $p_{\mathcal{Q}_j}$ calibrated by the matrix Frobenius norm in (51), and the conditional sampling probability $\overline{p}_{\mathcal{Q}_j}^f$ in (61) using multiple steps $f = 1, 2, 3$ are applied, where the corresponding convergence rates are denoted by ρ_{norm}, $\rho_{\text{Frobenius}}$, $\rho_{\text{Frobenius},f=1}$, $\rho_{\text{Frobenius},f=2}$, $\rho_{\text{Frobenius},f=3}$ respectively. Clearly, the convergence gain of using the Frobenius norm-based sampling distribution over the uniform sampling distribution can be readily confirmed. Based on it, improved convergence gains are attainted by the usage of multi-step conditional sampling. Specifically, the convergence of RBI-TPC is expedited gradually with the increment of f, which is in accordance with the result of Corollary 3. On the other hand, as expected, it is clear that the convergence rates of all the schemes improve with the increase of the block size $q = 2, 4, 8, 16$. More precisely, as shown in Remark 1, for $q = 16$ the proposed OBI-TPC algorithm is actually an approximation of RBI-TPC using $\overline{p}_{\mathcal{Q}_j}^f$ and $f = 3$, which exhibits an improved convergence performance over the others.[2]

The proposed RBI-TPC and OBI-TPC are developed based on the traditional GS iterative methods, where the block-based and the ordered updates have been adopted for improving the convergence performance. Here, we point out that the blocked update can also be incorporated into the traditional Jacobi iteration in a similar way. Since the component updates in the Jacobi iterations only rely on the results of the previous iteration, there is no need to optimize the updating order. To this end, the Jacobi iteration lends itself to convenient parallel implementation over multiple block updates at the same time. Nevertheless, because the convergence of the Jacobi iterations is not as rapid as that of the GS iterations, the related blocked update is omitted here.

In contrast to the traditional gradient descent algorithms, there is no need to find the optimized step size during the iterations of the proposed RBI-TPC or OBI-TPC. Moreover,

[2]Similarly, for $K = 64$ and $q = 8$, OBI-TPC is an approximation of RBI-TPC using $\overline{p}_{\mathcal{Q}_j}^f$ and $f = 7$, which is not shown in Fig.1.

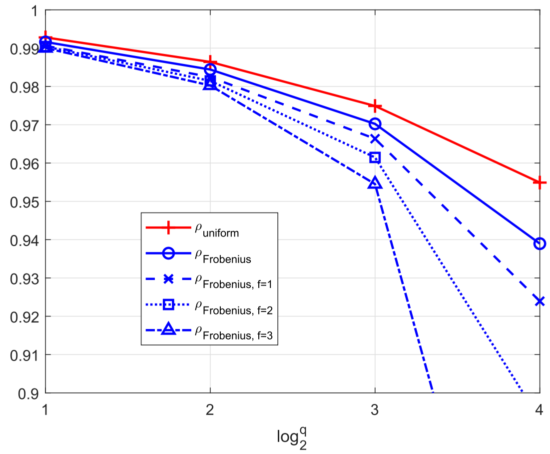


Fig. 1. Convergence comparison of RBI-TPC with different sampling probabilities in a 256×64 massive MIMO system.

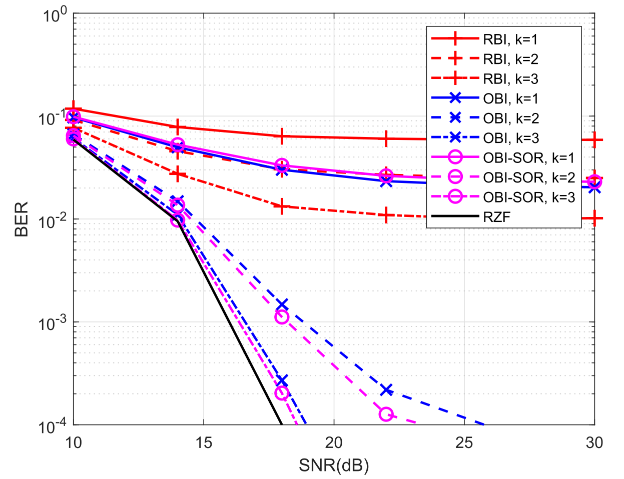


Fig. 2. BER versus average SNR for 128×32 massive MIMO systems with 16-QAM.

in principle, the proposed TPC schemes belong to the family of coordinate descent algorithms, since they update some components but leave the other components unchanged at the same time. More specifically, stemming from GS-TPC, RBI-TPC operates as a variant of randomized block coordinate descent algorithm [50-52], which is designed for linear precoding transmission in the downlink of massive MIMO systems. Different from RBI-TPC, inspired by multi-step conditional sampling, OBI-TPC behaves as a deterministic version of block coordinate descent algorithm but with an optimized updating order. As a result, the advantages of OBI-TPC over RBI-TPC in terms of convergence and efficiency are confirmed by both our theoretical and simulation results, making OBI-TPC more appealing for implementation in practice.

C. Further Enhancement by SOR-Aided Iteration

To further enhance the convergence, based on GS iteration, successive over-relaxation (SOR) iteration is proposed by applying a relaxation factor of $1 < \omega < 2$ in the form of [49]

$$(\mathbf{D} + \omega\mathbf{U})\mathbf{t}^{k+1} = [(1-\omega)\mathbf{D} - \omega\mathbf{E}]\mathbf{t}^k + \omega\mathbf{s}. \tag{70}$$

We note that GS iteration may also be viewed as a special case of SOR iteration associated with $\omega = 1$. It has been shown in [52] that SOR iteration achieves a faster convergence than GS iteration given a good choice of ω.

In both Jacobi and GS iterations, the result t_i^k is not considered in the update t_i^{k+1} (this can be verified more straightforward from (13) and (15)). By contrast, the SOR iteration exploits t_i^k in a beneficial way. In particular, the update of t_i^{k+1} in the SOR iteration obeys

$$\begin{aligned} t_i^{k+1} &= t_i^k + \frac{\omega}{a_{i,i}}\left(s_i - \mathbf{A}_{:,i}^H\bar{\mathbf{t}}^k\right) \\ &= t_i^k + \frac{\omega}{a_{i,i}}\left(s_i - \sum_{j=1}^{i} a_{i,j}t_j^k - \sum_{j=i+1}^{K} a_{i,j}t_j^{k+1}\right) \\ &= (1-\omega)t_i^k + \frac{\omega}{a_{i,i}}\left(s_i - \sum_{j=1}^{i-1} a_{i,j}t_j^k - \sum_{j=i+1}^{K} a_{i,j}t_j^{k+1}\right) \\ &= (1-\omega)t_i^k + \frac{\omega}{a_{i,i}}\left(s_i - \mathbf{A}_{:,i}^H\widehat{\mathbf{t}}^k\right) \end{aligned} \tag{71}$$

in conjunction with $\widehat{\mathbf{t}}^k = [t_1^k, \ldots, t_{i-1}^k, 0, t_{i+1}^{k+1}, \ldots, t_K^{k+1}]^H$, where the impact of t_i^k upon updating t_{i+1}^k is flexibly controlled by the relaxation factor of $1 < \omega < 2$. Again, for $\omega = 1$, the SOR iteration degenerates to the GS iteration, hence the result of t_i^k is ignored in updating t_i^{k+1}.

As shown in (71), in addition to $[t_1^k, \ldots, t_i^k, t_{i+1}^{k+1}, \ldots, t_K^{k+1}]$, the result t_i^k is also employed in the SOR iteration to update the element t_i^{k+1}, where again the traditional GS iteration may be viewed as a special case of SOR iteration using $\omega = 1$. Motivated by this, we can further upgrade the proposed RBI-TPC and OBI-TPC to the related SOR versions as follows

$$\mathbf{t}^{l+1} = \mathbf{t}^l + \omega\mathbf{I}_{:,\mathcal{Q}_j}(\mathbf{H}_{:,\mathcal{Q}_j}^H\mathbf{H}_{:,\mathcal{Q}_j})^{-1}\mathbf{I}_{:,\mathcal{Q}_j}^H\left(\mathbf{s} - \mathbf{A}\mathbf{t}^l\right) \tag{72}$$

with $1 < \omega < 2$. By doing this, the current block elements $\mathbf{t}_{\mathcal{Q}_j}$ will also be considered in the next update of $\mathbf{t}_{\mathcal{Q}_j}$, thus attaining an improved convergence performance. As for the choice of the relaxation parameter ω, the solution $\omega = \frac{2}{1+\sqrt{1-[\mu(\mathbf{I}-\mathbf{D}^{-1}\mathbf{A})]^2}}$ provided in [52] is only suited to the case $N \gg K$ while $\omega = 1.05$ is recommended experimentally.

VI. Simulations

In this section, the algorithms conceived for the massive MIMO downlink are evaluated by simulations. Specifically, the Gaussian fading environment having a perfectly known channel matrix $\mathbf{H} \in \mathbb{C}^{N\times K}$ at the base station is assumed. For a fair comparison with other iterative TPC schemes, the full iteration indexed by k is applied in both RBI-TPC and OBI-TPC schemes with the initial setup of $\mathbf{t}^0 = \mathbf{D}^{-1}\mathbf{s}$.

In Fig. 2, the performance of the proposed RBI-TPC and OBI-TPC schemes is shown in terms of the bit error rate (BER). In particular, a 128×32 massive MIMO system is applied using 16-QAM while the block size of $q = 8$ is employed in both RBI-TPC and OBI-TPC. As it is clearly seen, with the iteration index k, the BER of both RBI-TPC and OBI-TPC improves gradually, which is accordance with the reasons of Theorem 1 & 2. Meanwhile, given a specific iteration index k, we can observe that OBI-TPC has a better BER performance than RBI-TPC. This is because OBI-TPC achieves a faster convergence than RBI-TPC as a benefit of its

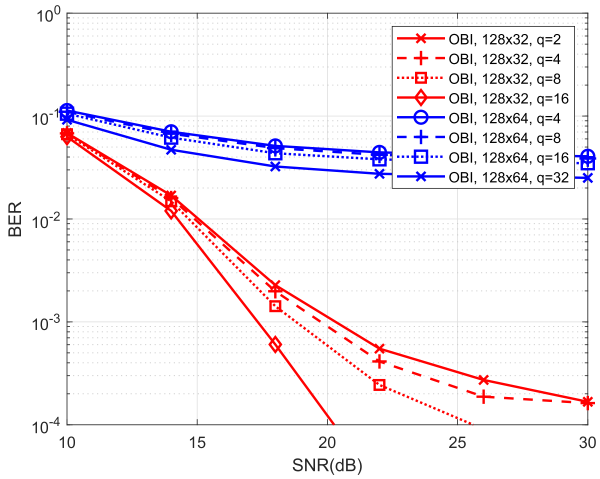


Fig. 3. BER versus average SNR for massive MIMO systems with 16-QAM.

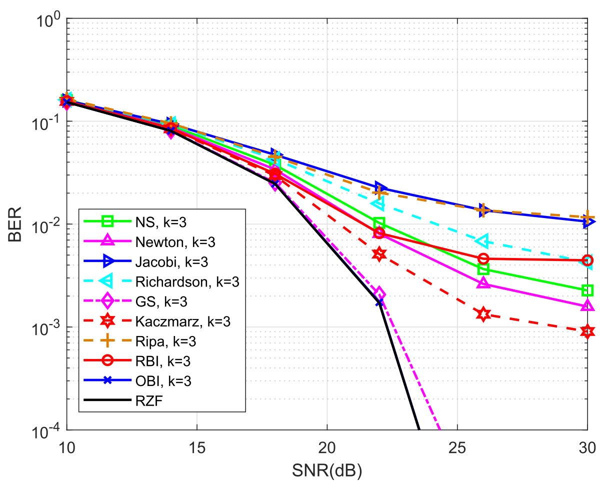


Fig. 4. BER versus average SNR for 128×16 massive MIMO systems with 64-QAM.

$r-1$ step conditional sampling. The related proof can be found in Corollary 3. On the other hand, motivated by SOR iteration, by introducing the relaxation parameter of $\omega = 1.05$ into OBI-TPC, the proposed OBI-SOR-TPC attains a better BER than OBI-TPC. More specifically, with iteration index $k = 3$, near-RZF TPC performance is realized by both OBI-TPC and OBI-SOR-TPC schemes.

In Fig. 3, the impact of the block size q is investigated in both 128×32 and 128×64 massive MIMO systems using 16-QAM, where different choices of q are applied in the proposed OBI-TPC for $k = 2$. Clearly, the BER performance of OBI-TPC in 128×32 and 128×64 massive MIMO systems improves gradually with the increase of the block size q. This is because a higher q means that more components of $\mathbf{t}$ will be updated at a time, thus resulting in a faster convergence. Nevertheless, according to the iteration in (69), a large size q also results in a higher complexity. Therefore, the block size q controls the TPC trade-off between performance and complexity in the downlink of massive MIMO.

In Fig. 4, the BER of the proposed TPC schemes is compared to that of the traditional iterative TPC schemes for a 128×16 massive MIMO system using 64-QAM. In addition

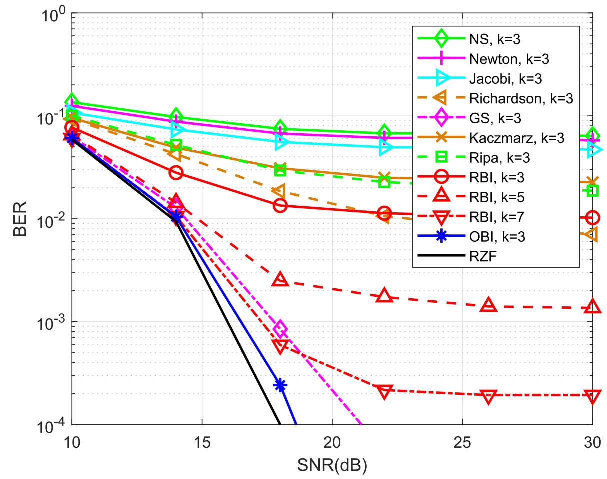


Fig. 5. BER versus average SNR for 128×32 massive MIMO systems with 16-QAM.

to RBI-TPC and OBI-TPC with $q = 8$, Neumann series (NS), Jacobi iteration, Newton iteration, Richardson iteration, Gauss-Seidel iteration, random Kaczmarz iterations in [46] and randomized iteration precoding algorithm (RIPA) in [37] are applied. Furthermore, the RZF TPC serves as a performance benchmark. Since the number of antennas at the base station is much higher than that at the user side, i.e., $N \gg K$, the convergence of the traditional iterative TPC schemes like NS series, Newton iteration, and Jacobi iteration is readily ensured. As seen in Fig. 4, under the same iteration index $k = 3$, OBI-TPC achieves the best BER performance among these TPC schemes, and the exact RZF TPC performance can be approximated by it. Compared to the traditional GS iteration, higher convergence gains can be exploited by OBI-TPC as a benefit of its more sophisticated updates, making OBI-TPC a better choice for downlink TPC.

In Fig. 5, the BER performance comparison of RBI-TPC and OBI-TPC is shown for massive MIMO systems associated with $N = 128$ and $K = 32$ using 16-QAM. In this case, since the convergence requirement of $N \gg K$ is not fulfilled, the convergence of Neumann series, Newton iteration, and Jacobi iteration can not be guaranteed, thus resulting in poor BER performance. By contrary, due to the global convergence transpiring from Theorem 1 & 2, both RBI-TPC and OBI-TPC perform well despite the increase of K. Clearly, as K increases from 16 to 32, more iterations are needed for all iterative TPC schemes due to the high system dimension. Here, the BER performance of RBI-TPC using $k = 3, 5, 7$ and OBI-TPC with $k = 3$ are given respectively. We can observe that OBI-TPC outperforms all the other iterative TPC schemes under $k = 3$, and near-RZF TPC performance is achieved upon increasing the number of iterations.

As a counterpart of Fig. 5, Fig. 6 shows the BER performance of the proposed RBI-TPC and OBI-TPC schemes for imperfect channel state information (CSI) in a 128×32 massive MIMO systems using 16-QAM. Explicitly, we let $\widehat{\mathbf{H}} = \mathbf{H} + \Delta\mathbf{H}$ represent the channel matrix for an imperfect CSI. Here, $\Delta\mathbf{H}$ represents the errors of channel estimation while each of its elements obeys $\mathcal{CN}(0, \sigma_e^2)$

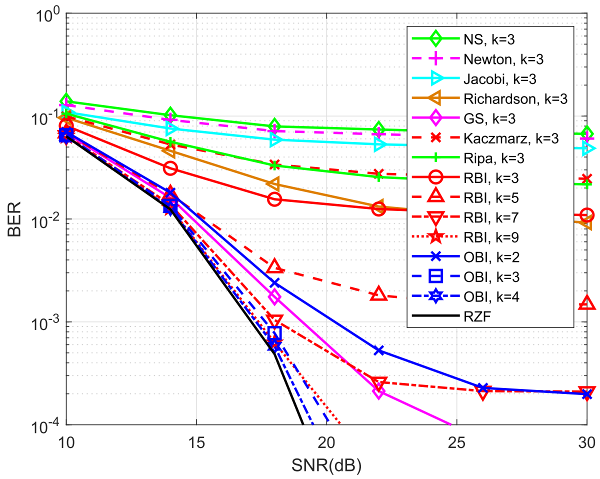


Fig. 6. BER versus average SNR for 128×32 massive MIMO systems with 16-QAM under imperfect CSI.

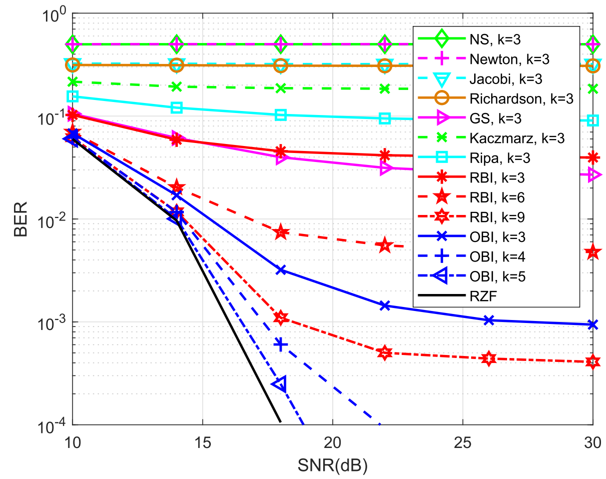


Fig. 7. BER versus average SNR for 128×32 massive MIMO systems with 16-QAM and correlation index $\psi = 0.05$.

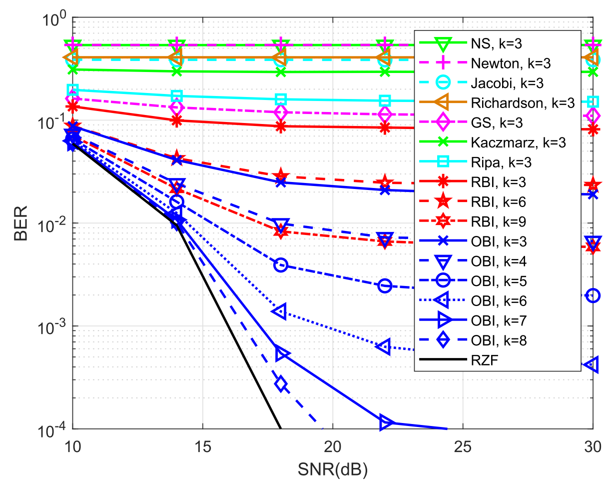


Fig. 8. BER versus average SNR for 128×32 massive MIMO systems with 16-QAM and correlation index $\psi = 0.1$.

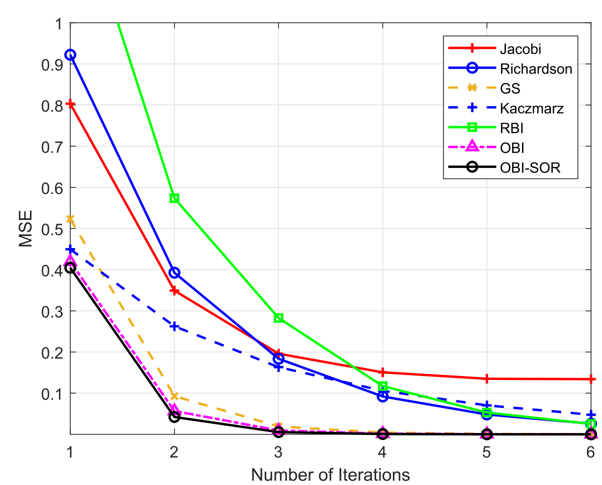


Fig. 9. MSE versus number of full iterations for 128×32 massive MIMO systems with fixed SNR at 18 dB.

with $\sigma_e^2 = 0.1$ [53], [54]. For imperfect CSI, the BER of all the TPC schemes degrades compared to the results of Fig. 4. Nevertheless, compared to other iterative TPC schemes the proposed OBI-TPC has the edge and near-RZF TPC performance can be attained upon increasing k. More specifically, according to (29) and (60) in Theorems 1 & 2, both the proposed RBI-TPC and OBI-TPC schemes are capable of global convergence. This means no extra requirements must be satisfied to guarantee the convergence. Therefore, they can work well in various cases of interest, which is not the case for traditional iterative schemes.

Apart from the independent, identically distributed (i.i.d.) Gaussian fading channels, here the impact of correlated fading channels is also investigated to illustrate the BER performance of the proposed RBI-TPC and OBI-TPC schemes for $q = 8$. In particular, following the configurations of the correlated fading channels in [55] and [56], the correlated fading channel matrix is set as $\mathbf{R}_{\mathrm{b}}^{\frac{1}{2}}\mathbf{H}\mathbf{R}_{\mathrm{u}}^{\frac{1}{2}}$, where $\mathbf{R}_{\mathrm{b}} \in \mathbb{C}^{N\times N}$ and $\mathbf{R}_{\mathrm{u}} \in \mathbb{C}^{K\times K}$ are the correlation matrices at two sides of the transceiver. Briefly, the normalized correlation coefficient $1 \geq \psi \geq 0$ determines the degree of correlation in these fading channels, where $\psi = 0$ accounts for the uncorrelated channel matrix and $\psi = 1$ for a fully correlated one. Specifically, compared to i.i.d. case in Fig. 5, the precoding performance of RZF is slightly eroded for the normalized correlation index of $\psi = 0.05$ in Fig. 7. However, the BER of the traditional iterative TPC schemes relying on Neumann series, Newton iteration, Jacobi iteration, Richardson iteration become poor since their convergence suffers from the correlated fading channels. Essentially, this is because a more correlated fading channel generally has a higher *condition number*, which is harmful to their convergence.

In contrast to the traditional iterative TPC schemes, both the proposed RBI-TPC and OBI-TPC still perform adequately but at a slower convergence. Specifically, their precoding performance gradually improve upon increasing k, which is in line with the convergence results of Theorem 1 & 2. We claim that at the same $k = 3$, the proposed OBI-TPC achieves the best BER performance. Similar results can also be found in Fig. 8, where the channel fading matrix turns out to be more correlated when the normalized correlation index is set to $\psi = 0.1$. Given the ill-condition of the channel fading

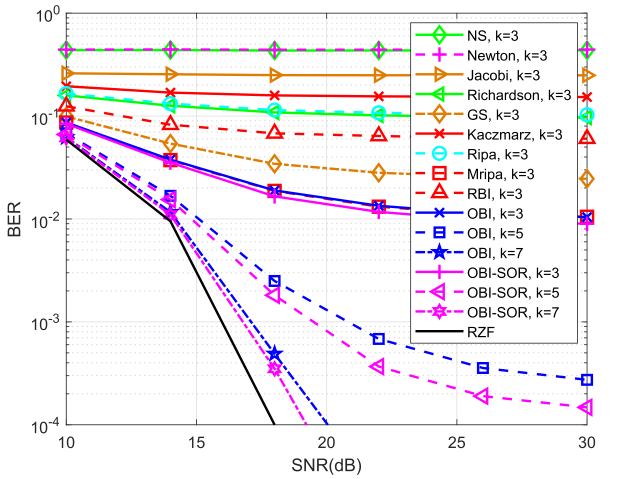


Fig. 10. BER versus average SNR for 128 × 64 massive MIMO systems with 16-QAM.

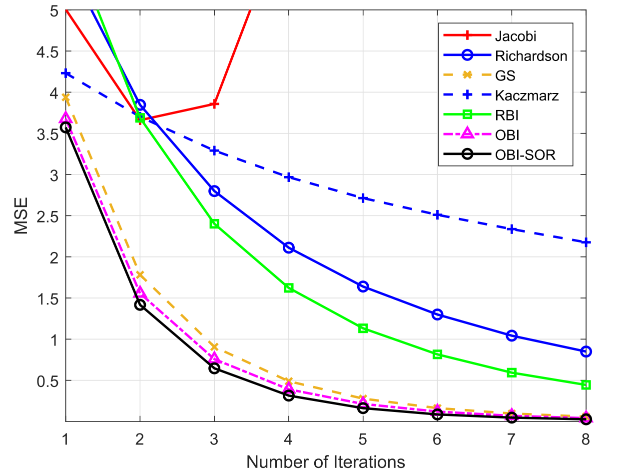


Fig. 11. MSE versus number of full iterations for 128 × 64 massive MIMO systems with fixed SNR at 18 dB.

matrix, the BER performance of the RZF TPC also becomes worse than before. Note that in this case the traditional iterative TPC schemes using Neumann series, Jacobi, Newton, Richardson do not work any more due to the poor convergence properties. In sharp contrast to them, the convergence of RBI-TPC and OBI-TPC is guaranteed, but more iterations are required for approaching the near-RZF TPC performance. Clearly, considerable performance gains can be obtained by the proposed OBI-TPC, making it eminently suitable for massive MIMO systems.

In Fig. 9, the convergence comparison among various iterative TPC schemes is presented for a 128×32 massive MIMO system using 16-QAM at a fixed SNR of 18 dB, where the mean square error (MSE) between the RZF TPC (i.e., $\mathbf{G}_{\mathrm{rzf}}\mathbf{s}$) and other TPC schemes (i.e., $\beta\mathbf{H}\mathbf{t}^k$) is used as the convergence criterion. As shown in (5), the linear precoding matrix $\mathbf{G}_{\mathrm{rzf}}$ can be equivalently acquired by obtaining the vector $\mathbf{t}$ in the linear system. Specifically, the MSE is gradually reduced upon increasing the number of full iterations from $k = 1$ to $k = 6$. A lower MSE under the same k implies a faster convergence, which naturally leads to a better TPC performance. As clearly seen, the proposed OBI-TPC achieves more rapid convergence than the traditional iterative TPC schemes - such as the Jacobi, Richardson, Kaczmarz and GS arrangements - where a near-RZF performance can be obtained by OBI-TPC along with $k = 3$. Based on this, a better convergence is achieved by OBI-SOR-TPC, which results in an improved TPC performance in massive MIMO systems. Overall, the convergence performance of all the TPC schemes is in line with the TPC performance shown in Fig. 4.

In Fig. 10, the precoding performance comparison of 128×64 massive MIMO systems using 16-QAM is presented. Compared to 128×16 and 128×32 scenarios, the antenna ratio N/K between the two sides of the channel becomes smaller. In this scenario, traditional Neumann series, Jacobi iteration, Newton iteration, and Richardson iteration fail to meet the convergence requirement of $N \gg K$. On the other hand, thanks to its global convergence, the proposed OBI-TPC still works well, and its BER performance improves gradually with the number of iterations. In addition, we can observe that MRIPA in [37] has comparable BER with OBI-TPC. Nevertheless, as discussed before, the complexity of OBI-TPC (i.e., $O(q^2K + K^2)$) is much less than that of MRIDA (i.e., $O(q^2K + 4K^2)$). Here, the BER performance of the proposed OBI-SOR-TPC is also shown, which outperforms the standard OBI-TPC by adopting the relaxation parameter of $\omega = 1.05$. Therefore, compared to OBI-TPC, OBI-SOR-TPC is preferable in practice.

In Fig. 11, the convergence comparison of various iterative TPC schemes is illustrated for a 128×64 massive MIMO system using 16-QAM with fixed SNR at 18 dB, where the MSE between the RZF TPC (i.e., $\mathbf{G}_{\mathrm{rzf}}\mathbf{s}$) and other TPC schemes (i.e., $\beta\mathbf{H}\mathbf{t}^k$) is used as the convergence criterion. Note that the Jacobi iteration fails to perform well in this scenario, since its MSE does not converge as well as that of the other TPC schemes. Except for the Jacobi iterations, the MSE of the TPC schemes decays gradually upon increasing the number of iterations from $k = 1$ to $k = 8$. As expected, the proposed OBI-SOR-TPC scheme has the best convergence performance for a given number of iterations, followed by the proposed OBI-TPC scheme. More specifically, near-RZF performance can be attained by them for $k = 8$. Clearly, the convergence performance directly determines the related TPC performance in massive MIMO systems, and the convergence performance of all the TPC schemes is in line with the TPC performance shown in Fig. 10.

In Table II, the complexity comparison of various TPC schemes for $128 \times K$ massive MIMO is illustrated in terms of the average number of flops per iteration. Specifically, since the complexity of NS-based TPC increases dramatically when $k > 2$, the flops of its iterations for $k \leq 2$ are considered. Meanwhile, the computational flops of the traditional RZF TPC is also shown as a baseline. Intuitively, with the increment of K, the complexities of all the TPC schemes become higher accordingly. In contrast to RZF having a rapid complexity growth, the iterative TPC schemes have a slower complexity increment, which is on the complexity order of $O(K^2)$. Here, the proposed OBI-TPC associated with $q = 4$ is applied,

TABLE II
ILLUSTRATION OF THE COMPLEXITY COMPARISON BY MEANS OF FLOPS FOR $128 \times K$ MASSIVE MIMO

	$K=32$	$K=40$	$K=48$	$K=56$	$K=64$
RZF	32768	64000	110592	175616	262144
NS	3040	4760	6864	9352	12224
Jacobi	960	1520	2208	3024	3968
Richardson	992	1560	2256	3080	4032
GS	1056	1640	2352	3192	4160
SOR	1088	1680	2400	3248	4224
OBI	1536	2240	3072	4032	5120

which has a slightly higher complexity than the GS and SOR schemes. This is straightforward to understand because only a few block operations are incurred in OBI-TPC, which in turn leads to a faster convergence. Clearly, there is a latent TPC trade-off between performance and complexity in OBI-TPC based on simply tuning q, making it flexible for different wireless networks in practice. Exploiting this flexible trade-off for the proposed OBI-TPC will be one of our future goals.

VII. CONCLUSION

In this paper, we conceived sophisticated updating mechanisms for the traditional low-complexity iterative TPC to boost the convergence for improving the TPC performance in the downlink of massive MIMO systems. After briefly revisiting the traditional iterative methods, the RBI-TPC algorithm was proposed, which can be viewed as a statistically motivated version of the block-based iterative TPC relying on a random updating order by sampling. Then, according to our convergence analysis, the proposed RBI-TPC algorithm was demonstrated to be globally and exponentially convergent. Meanwhile, the convergence gain attained by the block-based update over multiple components was confirmed explicitly and a well-designed sampling distribution was shown to expedite the convergence. Then by applying the conditional sampling, the updating order in RBI-TPC becomes deterministic, which was optimized as a fixed but ordered update for a faster convergence rate. Hence, as a further advance, the OBI-TPC algorithm was proposed by seamlessly incorporating the mechanisms of the ordered and the block-based updates into the iterations, which can be further enhanced by taking advantage of a relaxation factor.

REFERENCES

[1] I. Tomkos, D. Klonidis, E. Pikasis, and S. Theodoridis, “Toward the 6G network Era: Opportunities and challenges,” *IT Prof.*, vol. 22, no. 1, pp. 34–38, Jan. 2020.

[2] F. Tariq, M. R. A. Khandaker, K.-K. Wong, M. A. Imran, M. Bennis, and M. Debbah, “A speculative study on 6G,” *IEEE Wireless Commun.*, vol. 27, no. 4, pp. 118–125, 2020.

[3] W. Saad, M. Bennis, and M. Chen, “A vision of 6G wireless systems: Applications, trends, technologies, and open research problems,” *IEEE Netw.*, vol. 34, no. 3, pp. 134–142, May 2020.

[4] E. G. Larsson, O. Edfors, F. Tufvesson, and T. L. Marzetta, “Massive MIMO for next generation wireless systems,” *IEEE Commun. Mag.*, vol. 52, no. 2, pp. 186–195, Feb. 2014.

[5] X. You et al., “Towards 6G wireless communication networks: Vision, enabling technologies, and new paradigm shifts,” *Sci. china Inf. Sci.*, vol. 64, no. 1, Nov. 2020.

[6] F. Rusek et al., “Scaling up MIMO: Opportunities and challenges with very large arrays,” *IEEE Signal Process. Mag.*, vol. 30, no. 1, pp. 40–60, Jan. 2013.

[7] Y. Wu, C.-K. Wen, C. Xiao, X. Gao, and R. Schober, “Linear precoding for the MIMO multiple access channel with finite alphabet inputs and statistical CSI,” *IEEE Trans. Wireless Commun.*, vol. 14, no. 2, pp. 983–997, Feb. 2015.

[8] S. Yang and L. Hanzo, “Fifty years of MIMO detection: The road to large-scale MIMOs,” *IEEE Commun. Surveys Tuts.*, vol. 17, no. 4, pp. 1941–1988, 4th Quart., 2015.

[9] Z. Zhang, K. Long, A. V. Vasilakos, and L. Hanzo, “Full-duplex wireless communications: Challenges, solutions, and future research directions,” *Proc. IEEE*, vol. 104, no. 7, pp. 1369–1409, Jul. 2016.

[10] W. Zhang et al., “Widely linear precoding for large-scale MIMO with IQI: Algorithms and performance analysis,” *IEEE Trans. Wireless Commun.*, vol. 16, no. 5, pp. 3298–3312, May 2017.

[11] S. Zarei, W. H. Gerstacker, R. Weigel, M. Vossiek, and R. Schober, “Robust MSE-balancing hierarchical linear/Tomlinson–Harashima precoding for downlink massive MU-MIMO systems,” *IEEE Trans. Wireless Commun.*, vol. 17, no. 11, pp. 7309–7324, Nov. 2018.

[12] C. Ding, J.-B. Wang, H. Zhang, M. Lin, and J. Wang, “Joint MU-MIMO precoding and resource allocation for mobile-edge computing,” *IEEE Trans. Wireless Commun.*, vol. 20, no. 3, pp. 1639–1654, Mar. 2021.

[13] C. Feng, W. Shen, J. An, and L. Hanzo, “Weighted sum rate maximization of the mmWave cell-free MIMO downlink relying on hybrid precoding,” *IEEE Trans. Wireless Commun.*, vol. 21, no. 4, pp. 2547–2560, Apr. 2022.

[14] S. Lyu, Z. Wang, Z. Gao, H. He, and L. Hanzo, “Lattice-based mmWave hybrid beamforming,” *IEEE Trans. Commun.*, vol. 69, no. 7, pp. 4907–4920, Jul. 2021.

[15] J. Piao, K. Niu, J. Dai, and L. Hanzo, “Polar-precoding: A unitary finite-feedback transmit precoder for polar-coded MIMO systems,” *IEEE Trans. Veh. Technol.*, vol. 70, no. 11, pp. 12203–12208, Nov. 2021.

[16] Y. Chen, D. Chen, T. Jiang, and L. Hanzo, “Millimeter-wave massive MIMO systems relying on generalized sub-array-connected hybrid precoding,” *IEEE Trans. Veh. Technol.*, vol. 68, no. 9, pp. 8940–8950, Sep. 2019.

[17] J. Hoydis, S. ten Brink, and M. Debbah, “Massive MIMO in the UL/DL of cellular networks: How many antennas do we need?” *IEEE J. Sel. Areas Commun.*, vol. 31, no. 2, pp. 160–171, Feb. 2013.

[18] N. Fatema, G. Hua, Y. Xiang, D. Peng, and I. Natgunanathan, “Massive MIMO linear precoding: A survey,” *IEEE Syst. J.*, vol. 12, no. 4, pp. 3920–3931, Dec. 2018.

[19] T. Xie, L. Dai, X. Gao, X. Dai, and Y. Zhao, “Low-complexity SSOR-based precoding for massive MIMO systems,” *IEEE Commun. Lett.*, vol. 20, no. 4, pp. 744–747, Apr. 2016.

[20] Y. Liu, J. Liu, Q. Wu, Y. Zhang, and M. Jin, “A near-optimal iterative linear precoding with low complexity for massive MIMO systems,” *IEEE Commun. Lett.*, vol. 23, no. 6, pp. 1105–1108, Jun. 2019.

[21] A. Kammoun, A. Müller, E. Björnson, and M. Debbah, “Linear precoding based on polynomial expansion: Large-scale multi-cell MIMO systems,” *IEEE J. Sel. Topics Signal Process.*, vol. 8, no. 5, pp. 861–875, Oct. 2014.

[22] A. Benzin, G. Caire, Y. Shadmi, and A. M. Tulino, “Low-complexity truncated polynomial expansion DL precoders and UL receivers for massive MIMO in correlated channels,” *IEEE Trans. Wireless Commun.*, vol. 18, no. 2, pp. 1069–1084, Feb. 2019.

[23] C. Zhang, Y. Jing, Y. Huang, and L. Yang, “Performance analysis for massive MIMO downlink with low complexity approximate zero-forcing precoding,” *IEEE Trans. Commun.*, vol. 66, no. 9, pp. 3848–3864, Sep. 2018.

[24] J.-C. Chen, C.-J. Wang, K.-K. Wong, and C.-K. Wen, “Low-complexity precoding design for massive multiuser MIMO systems using approximate message passing,” *IEEE Trans. Veh. Technol.*, vol. 65, no. 7, pp. 5707–5714, Jul. 2016.

[25] D. Kwon, W.-Y. Yeo, and D. K. Kim, “A new precoding scheme for constructive superposition of interfering signals in multiuser MIMO systems,” *IEEE Commun. Lett.*, vol. 18, no. 11, pp. 2047–2050, Nov. 2014.

[26] B. Nagy, M. Elsabrouty, and S. Elramly, “Fast converging weighted Neumann series precoding for massive MIMO systems,” *IEEE Wireless Commun. Lett.*, vol. 7, no. 2, pp. 154–157, Apr. 2018.

[27] Y. Man, Z. Li, F. Yan, S. Xing, and L. Shen, “Massive MIMO pre-coding algorithm based on truncated Kapteyn series expansion,” in *Proc. IEEE Int. Conf. Commun. Syst. (ICCS)*, Dec. 2016, pp. 1–5.
[28] J. Minango and C. de Almeida, “A low-complexity linear precoding algorithm based on Jacobi method for massive MIMO systems,” in *Proc. IEEE 87th Veh. Technol. Conf. (VTC Spring)*, Jun. 2018, pp. 1–5.
[29] Z. Lu, J. Ning, Y. Zhang, T. Xie, and W. Shen, “Richardson method based linear precoding with low complexity for massive MIMO systems,” in *Proc. IEEE 81st Veh. Technol. Conf. (VTC Spring)*, May 2015, pp. 1–4.
[30] X. Gao, L. Dai, J. Zhang, S. Han, and I. Chih-Lin, “Capacity-approaching linear precoding with low-complexity for large-scale MIMO systems,” in *Proc. IEEE Int. Conf. Commun. (ICC)*, Jun. 2015, pp. 1577–1582.
[31] T. Xie, Q. Han, H. Xu, Z. Qi, and W. Shen, “A low-complexity linear precoding scheme based on SOR method for massive MIMO systems,” in *Proc. IEEE 81st Veh. Technol. Conf. (VTC Spring)*, May 2015, pp. 1–5.
[32] S. Zarei, W. Gerstacker, and R. Schober, “Low-complexity widely-linear precoding for downlink large-scale MU-MISO systems,” *IEEE Commun. Lett.*, vol. 19, no. 4, pp. 665–668, Apr. 2015.
[33] H. V. Nguyen, V.-D. Nguyen, and O.-S. Shin, “Low-complexity precoding for sum rate maximization in downlink massive MIMO systems,” *IEEE Wireless Commun. Lett.*, vol. 6, no. 2, pp. 186–189, Apr. 2017.
[34] Y. Man, C. Zhang, Z. Li, F. Yan, S. Xing, and L. Shen, “Massive MIMO pre-coding algorithm based on improved Newton iteration,” in *Proc. IEEE 85th Veh. Technol. Conf. (VTC Spring)*, Jun. 2017, pp. 1–5.
[35] P. Yang and H. Yang, “A low-complexity linear precoding for MIMO channels with finite constellation inputs,” *IEEE Wireless Commun. Lett.*, vol. 8, no. 5, pp. 1415–1418, Oct. 2019.
[36] M. A. Albreem, A. H. A. Habbash, A. M. Abu-Hudrouss, and S. S. Ikki, “Overview of precoding techniques for massive MIMO,” *IEEE Access*, vol. 9, pp. 60764–60801, 2021.
[37] Z. Wang, R. M. Gower, C. Zhang, S. Lyu, Y. Xia, and Y. Huang, “A statistical linear precoding scheme based on random iterative method for massive MIMO systems,” *IEEE Trans. Wireless Commun.*, vol. 21, no. 12, pp. 10115–10129, Dec. 2022.
[38] S. Liu, C. Ling, and D. Stehle, “Decoding by sampling: A randomized lattice algorithm for bounded distance decoding,” *IEEE Trans. Inf. Theory*, vol. 57, no. 9, pp. 5933–5945, Sep. 2011.
[39] P. Klein, “Finding the closest lattice vector when it’s unusually close,” in *Proc. 11th ACM-SIAM Symp. Discrete Algorithms*, 2000, pp. 937–941.
[40] Z. Wang and C. Ling, “Lattice Gaussian sampling by Markov chain monte carlo: Bounded distance decoding and trapdoor sampling,” *IEEE Trans. Inf. Theory*, vol. 65, no. 6, pp. 3630–3645, Jun. 2019.
[41] Z. Wang, S. Lyu, Y. Xia, and Q. Wu, “Expectation propagation-based sampling decoding: Enhancement and optimization,” *IEEE Trans. Signal Process.*, vol. 69, pp. 195–209, 2021.
[42] Z. Wang and C. Ling, “On the geometric ergodicity of metropolis-hastings algorithms for lattice Gaussian sampling,” *IEEE Trans. Inf. Theory*, vol. 64, no. 2, pp. 738–751, Feb. 2018.
[43] D. L. Donoho and J. Tanner, “Exponential bounds implying construction of compressed sensing matrices, error-correcting codes, and neighborly polytopes by random sampling,” *IEEE Trans. Inf. Theory*, vol. 56, no. 4, pp. 2002–2016, Apr. 2010.
[44] Z. Wang, L. Liu, and C. Ling, “Sliced lattice Gaussian sampling: convergence improvement and decoding optimization,” *IEEE Trans. Commun.*, vol. 69, no. 4, pp. 2599–2612, Apr. 2021.
[45] D. Zhu, B. Li, and P. Liang, “On the matrix inversion approximation based on Neumann series in massive MIMO systems,” in *Proc. IEEE Int. Conf. Commun. (ICC)*, Jun. 2015, pp. 1763–1769.
[46] M. N. Boroujerdi, S. Haghighatshoar, and G. Caire, “Low-complexity statistically robust precoder/detector computation for massive MIMO systems,” *IEEE Trans. Wireless Commun.*, vol. 17, no. 10, pp. 6516–6530, Oct. 2018.
[47] A. Björck, *Numerical Methods for Least squares Problems*. Philadelphia, PA, USA: SIAM, 1996.
[48] Y. Saad, *Iterative Methods for Sparse Linear Systems*. Philadelphia, PA, USA: SIAM, 1997.
[49] A. Greenbaum, *Iterative Methods for Solving Linear Systems*. Philadelphia, USA: SIAM, 1997.
[50] Y. Nesterov, “Efficiency of coordinate descent methods on huge-scale optimization problems,” *SIAM J. Optim.*, vol. 22, no. 2, pp. 341–362, Jan. 2012.
[51] P. Richtárik and M. Takáč, “Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function,” *Math. Program.*, vol. 144, nos. 1–2, pp. 1–38, Apr. 2014.
[52] A. Björck, *Numerical Methods in Matrix Computations*. Berlin, Germany: Springer, 2015.
[53] N. Lee, O. Simeone, and J. Kang, “The effect of imperfect channel knowledge on a MIMO system with interference,” *IEEE Trans. Commun.*, vol. 60, no. 8, pp. 2221–2229, Aug. 2012.
[54] D. L. Colon, F. H. Gregorio, and J. Cousseau, “Linear precoding in multi-user massive MIMO systems with imperfect channel state information,” in *Proc. 16th Workshop Inf. Process. Control (RPIC)*, Oct. 2015, pp. 1–6.
[55] H. R. Bahrami, T. Le-Ngoc, A. M. N. Nasrabadi, and S. H. Jamali, “Precoder design based on correlation matrices for MIMO systems,” in *Proc. IEEE Int. Conf. Commun. (ICC)*, vol. 3, May 2005, pp. 2001–2005.
[56] B. Costa, A. Mussi, and T. Abrão, “MIMO detectors under correlated channels,” *Semina, Ciencias Exatas e Tecnologicas*, vol. 37, no. 1, pp. 3–12, 2016.

Zheng Wang (Senior Member, IEEE) received the B.S. degree in electronic and information engineering from the Nanjing University of Aeronautics and Astronautics (NUAA), Nanjing, China, in 2009, the M.S. degree in communications from The University of Manchester, Manchester, U.K., in 2010, and the Ph.D. degree in communication engineering from Imperial College London, U.K., in 2015.

Since 2021, he has been an Associate Professor with the School of Information and Engineering, Southeast University, Nanjing. From 2015 to 2016, he was a Research Associate with Imperial College London. From 2016 to 2017, he was a Senior Engineer with the Radio Access Network Research and Development Division, Huawei Technologies Company. From 2017 to 2020, he was an Associate Professor with the College of Electronic and Information Engineering, NUAA. His current research interests include massive MIMO systems, machine learning and data analytics over wireless networks, and lattice theory for wireless communications.

Jiaheng Wang (Senior Member, IEEE) received the B.E. and M.S. degrees from Southeast University, Nanjing, China, in 2001 and 2006, respectively, and the Ph.D. degree in electronic and computer engineering from The Hong Kong University of Science and Technology, Hong Kong, in 2010.

From 2010 to 2011, he was with the Signal Processing Laboratory, KTH Royal Institute of Technology, Stockholm, Sweden. He also held visiting positions with the Friedrich Alexander University Erlangen-Nrnberg, Nürnberg, Germany, and the University of Macau, Macau. He is currently a Full Professor with the National Mobile Communications Research Laboratory (NCRL), Southeast University. He is also with Purple Mountain Laboratories, Nanjing. He has published more than 100 articles on international journals. His research interests include communication systems, wireless networks, and blockchain technologies.

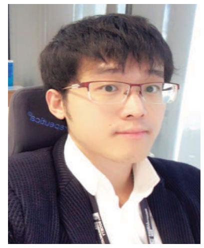
Dr. Wang was a recipient of the Humboldt Fellowship for Experienced Researchers. He received the Best Paper Awards of IEEE GLOBECOM 2019, ADHOCNETS 2019, WCSP 2014, and WCSP 2022. He serves as an Associate Editor for the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS and IEEE TRANSACTIONS ON COMMUNICATIONS and a Senior Area Editor for the IEEE SIGNAL PROCESSING LETTERS.

Zhen Gao received the B.S. degree in information engineering from the Beijing Institute of Technology, Beijing, China, in 2011, and the Ph.D. degree in communication and signal processing from the Tsinghua National Laboratory for Information Science and Technology, Department of Electronic Engineering, Tsinghua University, China, in 2016.

He is currently an Assistant Professor with the Beijing Institute of Technology. His research interests include wireless communications, with a focus on multi-carrier modulations, multiple antenna systems, and sparse signal processing. He was a recipient of IEEE Broadcast Technology Society 2016 Scott Helt Memorial Award (Best Paper), the Exemplary Reviewer of IEEE COMMUNICATIONS LETTERS in 2016, the *Electronics Letters* (IET) Premium Award (Best Paper) in 2016, and the Young Elite Scientists Sponsorship Program by the China Association for Science and Technology from 2018 to 2021.

Yongming Huang (Senior Member, IEEE) received the B.S. and M.S. degrees from Nanjing University, Nanjing, China, in 2000 and 2003, respectively, and the Ph.D. degree in electrical engineering from Southeast University, Nanjing, in 2007.

Since March 2007, he has been a Faculty Member of the School of Information Science and Engineering, Southeast University, where he is currently a Full Professor. From 2008 to 2009, he visited the Signal Processing Laboratory, Royal Institute of Technology, Stockholm, Sweden. He has authored or coauthored more than 200 peer-reviewed papers, and holds more than 80 invention patents. His research interests include intelligent 5G/6G mobile communications and millimeter wave wireless communications. He submitted around 20 technical contributions to IEEE standards. He was awarded a certificate of appreciation for outstanding contribution to the development of IEEE standard 802.11aj. He was an Associate Editor of the IEEE TRANSACTIONS ON SIGNAL PROCESSING and a Guest Editor of the IEEE JOURNAL SELECTED AREAS IN COMMUNICATIONS. He is currently an Editor-at-Large of the IEEE OPEN JOURNAL OF THE COMMUNICATIONS SOCIETY and an Associate Editor of the IEEE WIRELESS COMMUNICATIONS LETTERS.



Derrick Wing Kwan Ng (Fellow, IEEE) received the bachelor's degree (Hons.) and the Master of Philosophy (M.Phil.) degree in electronic engineering from The Hong Kong University of Science and Technology (HKUST) in 2006 and 2008, respectively, and the Ph.D. degree from The University of British Columbia (UBC) in November 2012. He was a Senior Post-Doctoral Fellow with the Institute for Digital Communications, Friedrich-Alexander-University Erlangen-Nürnberg (FAU), Germany. He is currently a Scientia Associate Professor with the University of New South Wales, Sydney, Australia. His research interests include global optimization, physical layer security, IRS-assisted communication, UAV-assisted communication, wireless information and power transfer, and green (energy-efficient) wireless communications.

Dr. Ng has been listed as a Highly Cited Researcher by Clarivate Analytics (Web of Science) since 2018. He received the Australian Research Council (ARC) Discovery Early Career Researcher Award in 2017, the IEEE Communications Society Leonard G. Abraham Prize in 2023, the IEEE Communications Society Stephen O. Rice Prize in 2022, the Best Paper Awards at the WCSP 2020, 2021, IEEE TCGCC Best Journal Paper Award in 2018, INISCOM 2018, IEEE International Conference on Communications (ICC) in 2018, 2021, and 2023, IEEE International Conference on Computing, Networking and Communications (ICNC) 2016, IEEE Wireless Communications and Networking Conference (WCNC) 2012, the IEEE Global Telecommunication Conference (GLOBECOM) 2011 and 2021, and the IEEE Third International Conference on Communications and Networking in China 2008. He served as an Editorial Assistant to the Editor-in-Chief for the IEEE TRANSACTIONS ON COMMUNICATIONS from January 2012 to December 2019. He is also serving as an Editor for the IEEE TRANSACTIONS ON COMMUNICATIONS and an Associate Editor-in-Chief for the IEEE OPEN JOURNAL OF THE COMMUNICATIONS SOCIETY.

Lajos Hanzo (Life Fellow, IEEE) received the Honorary Doctorates from the Technical University of Budapest in 2009 and Edinburgh University in 2015. He is currently a Foreign Member of the Hungarian Science-Academy, a fellow of the Royal Academy of Engineering (F.R.Eng.) of the IET and EURASIP, and holds the IEEE Eric Sumner Technical Field Award. For more information visit the link (http://www-mobile.ecs.soton.ac.uk/, https://en.wikipedia.org/wiki/Lajos_Hanzo).